Non-Hamiltonian and Non-Traceable Regular 3-Connected Planar Graphs

Nico VAN CLEEMPUT and Carol T. ZAMFIRESCU

By Euler's formula, there are k-regular 3-connected planar graphs for three values of k: 3, 4, or 5. Denote by c_k, resp. p_k, the order of the smallest non-hamiltonian, resp. non-traceable, k-regular 3-connected planar graph.

Tait conjectured in 1884 that every 3-regular 3-connected planar graph is hamiltonian. This conjecture became famous because it implied the Four Colour Theorem (which at that time was still the Four Colour Problem). However, Tait's conjecture turned out to be false and the first to construct a counterexample was Tutte in 1946. The smallest counterexample is due to Lederberg (and independently, Bosák and Barnette) and has order 38. That this is indeed the smallest possible counterexample was shown by Holton and McKay after a long series of papers by various authors, e.g., Butler, Barnette and Wegner, and Okamura. This settles that $c_3 = 38$. Combining work of Knorr and T. Zamfirescu, it has been shown that $54 \leq p_3 \leq 88$.

Non-hamiltonian 4-regular 3-connected planar graphs have been known for a long time. Following work of Walther and Sachs, Zaks proved that there exists a non-hamiltonian 4-regular 3-connected planar graph of order 209. Applying a technique by Sachs to transform a non-hamiltonian 3-regular 3-connected planar graph into a non-hamiltonian 4-regular 3-connected planar graph, Bosák showed that $c_4 \leq 171$. Using the same technique by Sachs and the non-traceable 3-regular 3-connected planar graph on 88 vertices by T. Zamfirescu, it can be shown that $p_4 \leq 396$.

Zaks showed for the 5-regular case that $c_5 \leq 532$ and that $p_5 \leq 1232$. Owens strongly improved these bounds for non-hamiltonian and non-traceable 5-regular 3-connected planar graphs. More specifically, he showed that $c_5 \leq 76$ and that $p_5 \leq 128$.

The main focus of the talk will be the 4-regular case for which we show that there exists a non-hamiltonian 4-regular 3-connected planar graph with 39 vertices. We also present the result of computations that show that every non-hamiltonian 4-regular 3-connected planar graph has at least 34 vertices. This implies that $34 \leq c_4 \leq 39$. Using similar building blocks, we also construct a non-traceable 4-regular 3-connected planar graph on 78 vertices which improves the upper bound of p_4 to $p_4 \leq 78$. Furthermore, we discuss a small improvement to the 5-regular non-traceble case by showing that $p_5 \leq 120$. For the 3-regular case, we turn our attention towards cyclically-4-edge-connected planar graphs.