Non-Hamiltonian and Non-Traceable Regular 3-Connected Planar Graphs

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- Here, a *polyhedron* is a planar 3-connected graph.
- The word "regular" is used exclusively in the graph-theoretical sense of having all vertices of the same degree.
- By Euler's formula, there are k-regular polyhedra for exactly three values of k: 3, 4, or 5.

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- Let c_k be the order of the smallest **non-hamiltonian** k-regular polyhedron.
- Let p_k be the order of the smallest **non-traceable** k-regular polyhedron.

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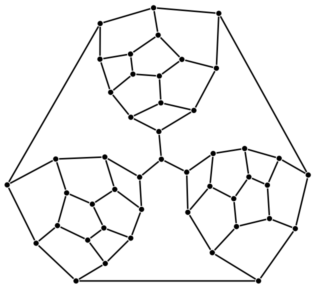
Cubic polyhedra – hamiltonicity

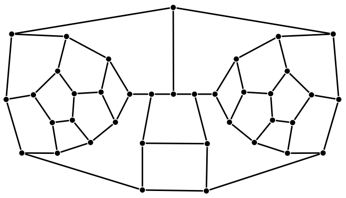
- Tait conjectured in 1884 that every cubic polyhedron is hamiltonian.
- The conjecture became famous because it implied the Four Colour Theorem (at that time still the Four Colour Problem)

Cubic polyhedra – hamiltonicity



The first to construct a counterexample (of order 46) was Tutte in 1946

Cubic polyhedra – hamiltonicity



Lederberg, Bosák, and Barnette (pairwise independently) described a smaller counterexample having 38 vertices.

Cubic polyhedra – hamiltonicity

After a long series of papers by various authors (e.g., Butler, Barnette, Wegner, Okamura), Holton and McKay showed that all cubic polyhedra on up to 36 vertices are hamiltonian.

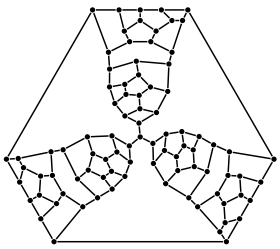
Theorem (Holton and McKay, 1988)

$c_3 = 38$

Cubic polyhedra – traceability

- Balinski asked whether cubic non-traceable polyhedra exist
- Brown and independently Grünbaum and Motzkin proved the existence of such graphs
- Klee asked for determining p_3

Cubic polyhedra – traceability



In 1970 T. Zamfirescu constructed this cubic non-traceable planar graph on 88 vertices

Cubic polyhedra – traceability

Based on work of Okamura, Knorr improved a result of Hoffmann by showing that all cubic planar graphs on up to 52 vertices are traceable.

Theorem (Knorr, 2010 and Zamfirescu, 1970)

$54 \leq p_3 \leq 88$

Quartic polyhedra – hamiltonicity

- Following work of Sachs from 1967 and Walther from 1969, Zaks proved in 1976 that there exists a quartic non-hamiltonian polyhedron of order 209.
- The actual number given in Zaks' paper is false, as pointed out in work of Owens — therein the correct number can be found.

Quartic polyhedra – hamiltonicity

Theorem (Sachs, 1967)

If there exists a non-hamiltonian (non-traceable) cubic polyhedron of order n, then there exists a non-traceable (non-hamiltonian) quartic polyhedron on $\frac{9n}{2}$ vertices.

On page 132 of Bosák's book it is claimed that converting the Lederberg-Bosák-Barnette graph with this method gives a quartic non-hamiltonian polyhedron of order 161. However, the correct number should be $38 \times \frac{9}{2} = 171$.

Theorem (Sachs, 1967 combined with Bosák, 1990)

$c_4 \leq 171$

Quartic polyhedra – traceability

- Zaks showed that $p_4 \leq 484$
- Using Sachs' theorem on Zamfirescu's 88-vertex graph gives a non-traceable quartic polyhedron on 396 vertices.

Theorem (Sachs, 1967 combined with Zamfirescu, 1980)

$p_4 \leq 396$

Quintic polyhedra

- Previous work includes papers by Walther, as well as Harant, Owens, Tkáč, and Walther.
- Zaks showed that $c_5 \leq 532$ and $p_5 \leq 1232$.
- Owens proved that $c_5 \leq 76$ and $p_5 \leq 128$.

Theorem (Owens, 1980)

$c_5 \leq 76$
$p_5 \leq 128$

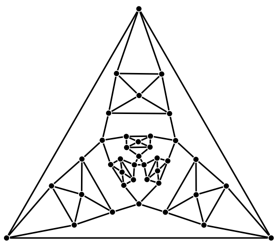
Summary

	Hamiltonicity	Traceability
Cubic	$c_3 = 38$	$54 \leq p_3 \leq 88$
Quartic	$c_4 \leq 171$	$p_4 \leq 396$
Quintic	$c_5 \leq 76$	$p_5 \leq 128$

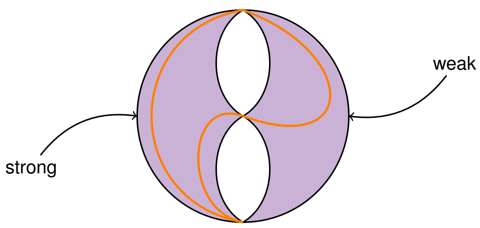
Upper bound hamiltonicity

Theorem (Van Cleemput and Zamfirescu, 2017+)

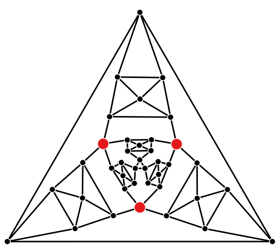
$c_4 \leq 39$

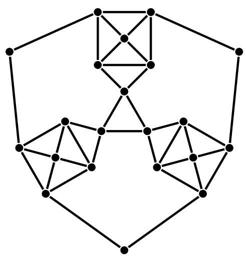


Upper bound hamiltonicity

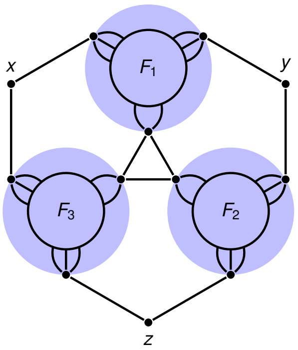


Upper bound hamiltonicity

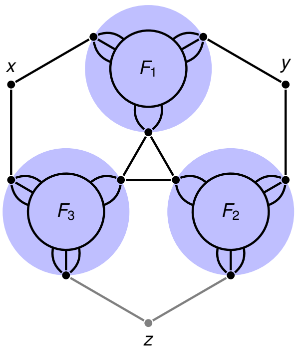




Upper bound hamiltonicity



Upper bound hamiltonicity

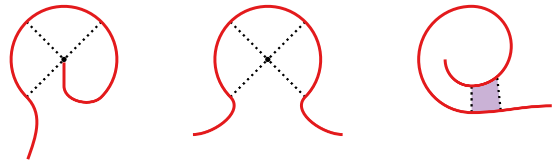


Lower bound hamiltonicity

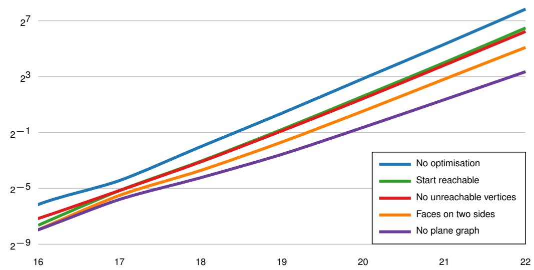
Check all quartic polyhedra for being hamiltonian.

Lower bound hamiltonicity – hamiltonicity check

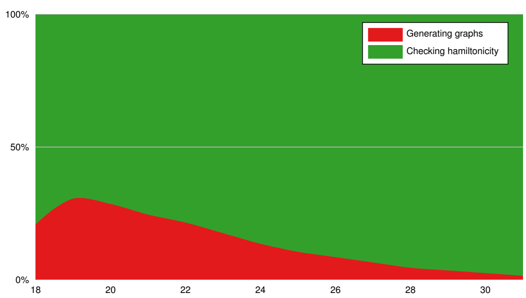
Simple backtracking algorithm that tries to construct a cycle from the first vertex.



Lower bound hamiltonicity – hamiltonicity check



Lower bound hamiltonicity – generation



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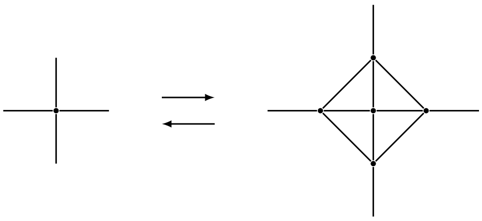

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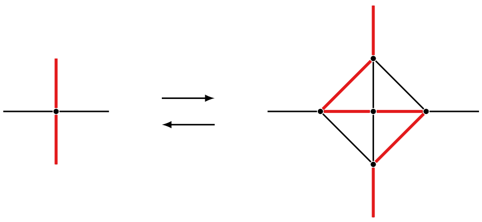

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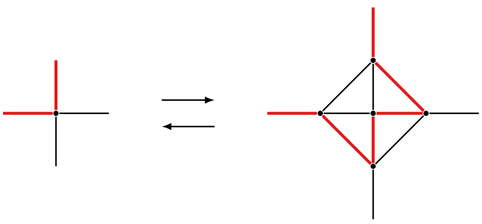

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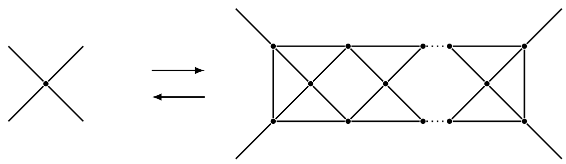

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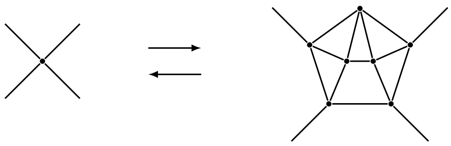

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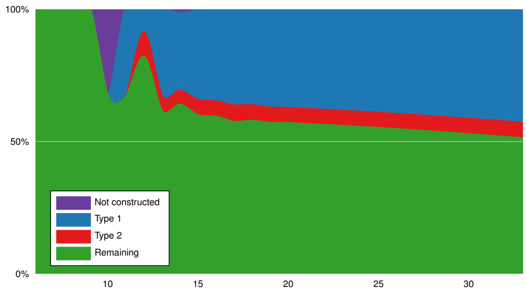

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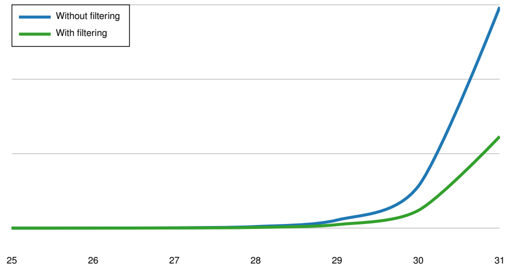


Lower bound hamiltonicity

Theorem (Van Cleemput and Zamfirescu, 2017+)

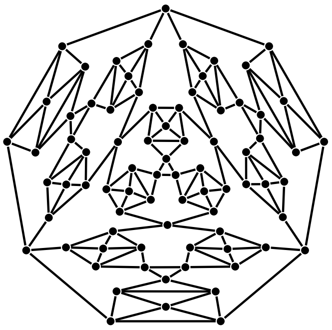
$c_4 \geq 34$

Vertices	Time
25	7.7 minutes
26	34.9 minutes
27	2.7 hours
28	13.2 hours
29	2.7 days
30	13.8 days
31	71.3 days
32	1.0 years
33	5.5 years

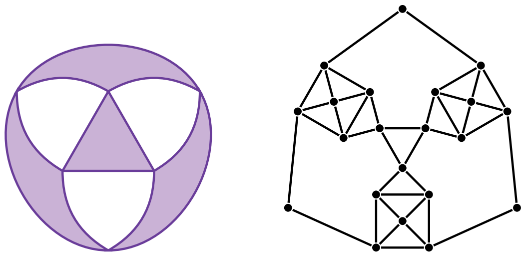
Upper bound traceability

Theorem (Van Cleemput and Zamfirescu, 2017+)

$p_4 \leq 78$



Upper bound traceability

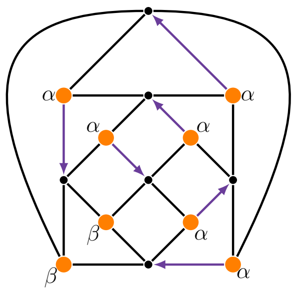


$21 \times 4 - 6 = 78$ vertices

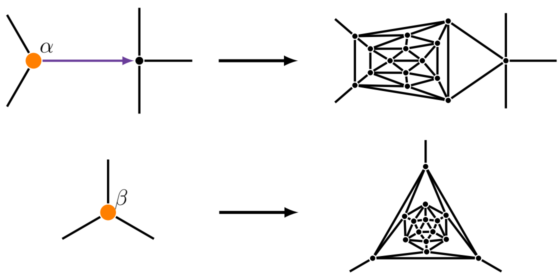
Upper bound traceability

Theorem (Van Cleemput and Zamfirescu, 2017+)

$p_5 \leq 120$



Upper bound traceability



Summary

	Hamiltonicity	Traceability
Cubic	$c_3 = 38$	$54 \leq p_3 \leq 88$
Quartic	~~$c_4 \leq 171$~~ $34 \leq c_4 \leq 39$	~~$p_4 \leq 396$~~ $34 \leq p_4 \leq 78$
Quintic	~~$c_5 \leq 76$~~ $38 \leq c_5 \leq 76$	~~$p_5 \leq 128$~~ $38 \leq p_5 \leq 120$

Future work

- $c_4 \geq 35$?
- Hamiltonicity for quintic case
- Lower bounds for traceability