Connections between decomposition trees of 3-connected plane triangulations and hamiltonian properties

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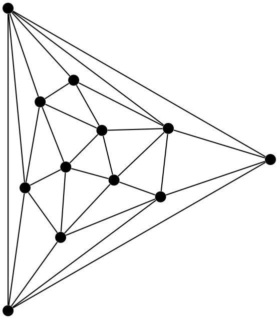
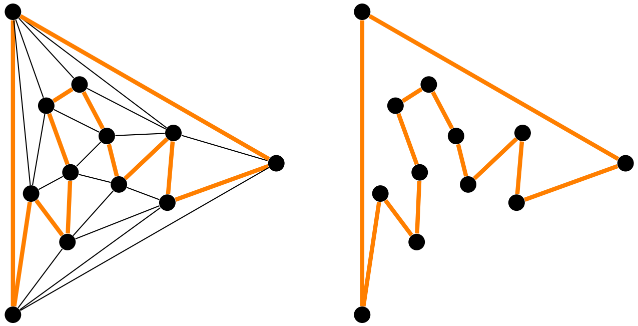
Triangulation

A triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

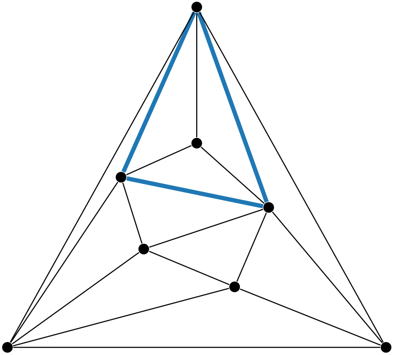
A hamiltonian cycle in $G(V,E)$ is a subgraph of $G(V,E)$ which is isomorphic to $C_{|V|}$.



A graph is hamiltonian if it contains a hamiltonian cycle.

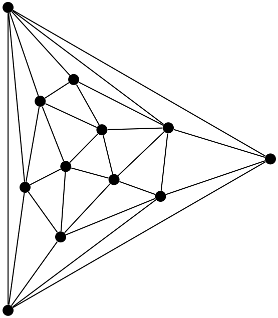
Separating triangles

A separating triangle S in a triangulation T is a subgraph of T such that S is isomorphic to C_3 and $T - S$ has two components.



4-connected triangulations

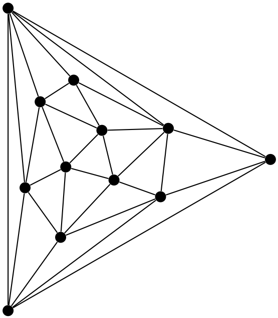
A triangulation is 4-connected if and only if it contains no separating triangles.



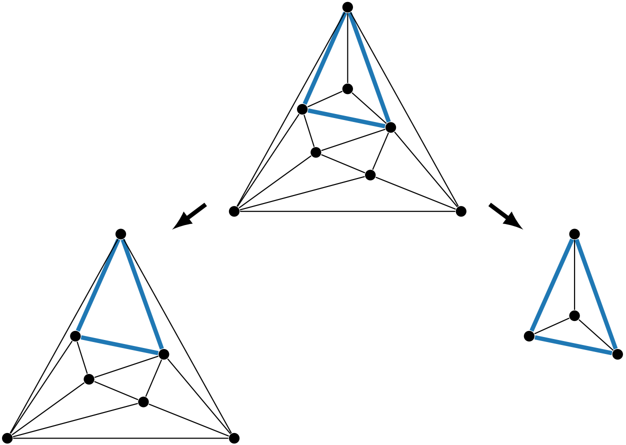
Whitney

Theorem (Whitney, 1931)

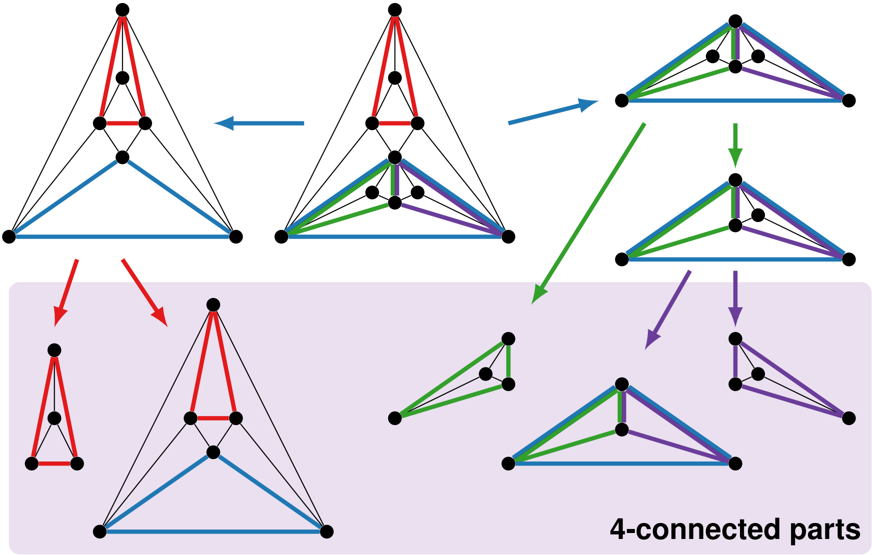
Each triangulation without separating triangles is hamiltonian.



Splitting triangulations

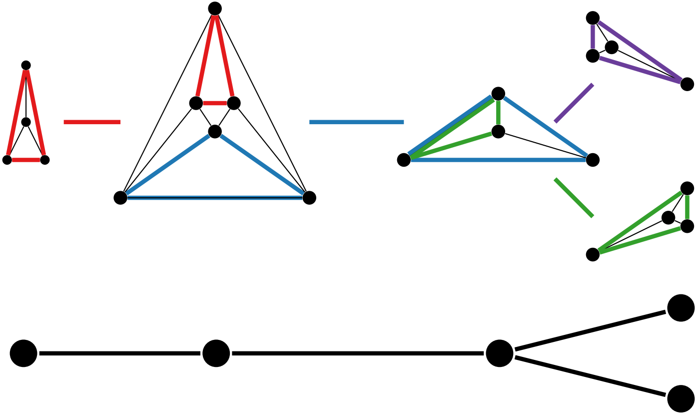


Recursively splitting triangulations



Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles



Decomposition trees and hamiltonicity

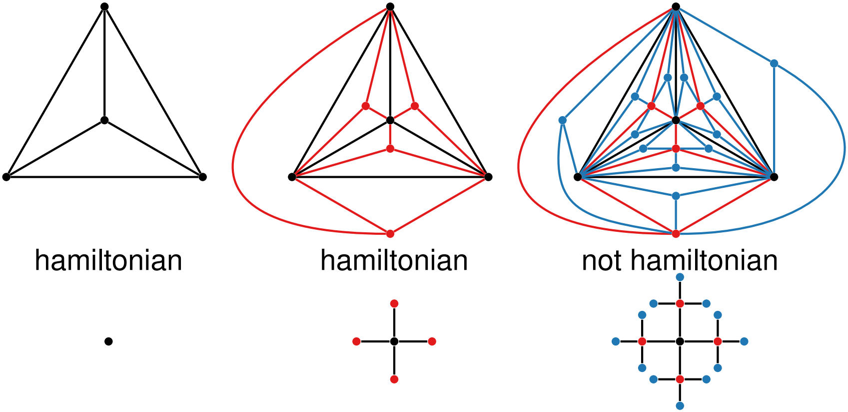
For each tree T there exist hamiltonian triangulations which have T as decomposition tree.

A triangulation G with decomposition tree T is hamiltonian if ...

- Whitney (1930): $|E(T)| = 0$
- Thomassen (1978), Chen (2003): $|E(T)| \leq 1$
- Böhme, Harant, Tkáč (1993): $|E(T)| \leq 2$
- Jackson, Yu (2002): $\Delta(T) \leq 3$

Jackson and Yu

$\Delta(T) \leq 4$ is not sufficient to imply hamiltonicity.

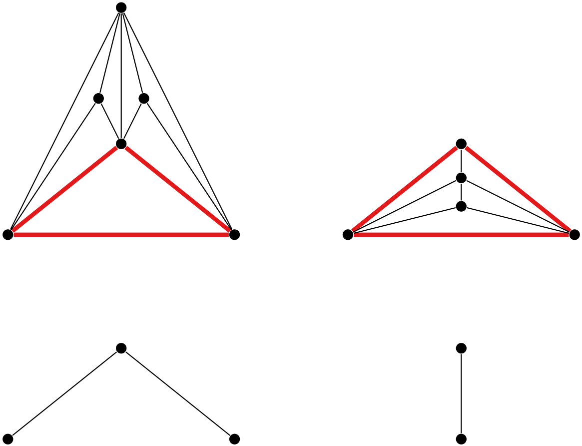


Question

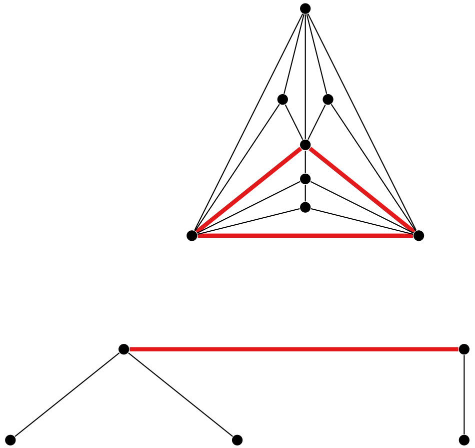
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?

Subdividing a face with a graph



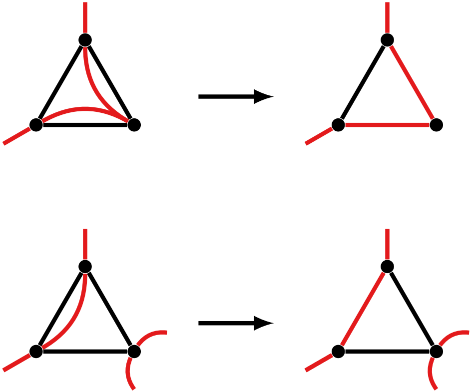
Subdividing a face with a graph



Subdividing a non-hamiltonian triangulation

Lemma

When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.

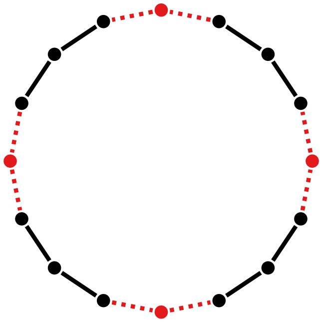


Toughness

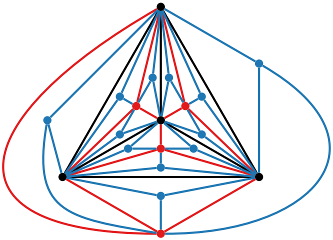
A graph is 1-tough if it cannot be split into k components by removing less than k vertices.

Toughness

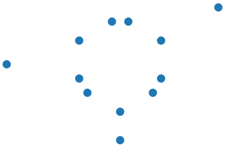
A hamiltonian graph is 1-tough.



Graphs that are not 1-tough are trivially non-hamiltonian.



Remove 4 black and 4 red vertices

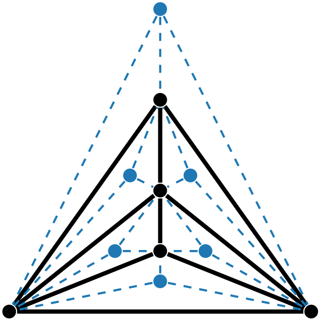


12 blue components remain

Creating a non-hamiltonian plane graph

Lemma

When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.



The subdivided graph is not 1-tough.

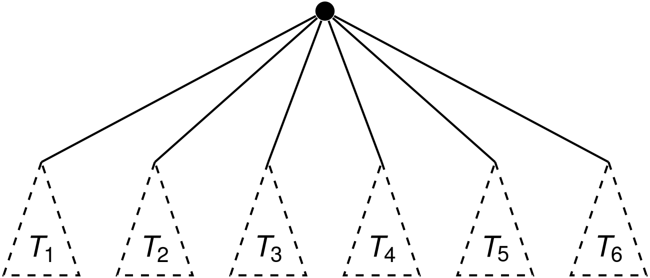
Decomposition trees with $\Delta \geq 6$

Theorem

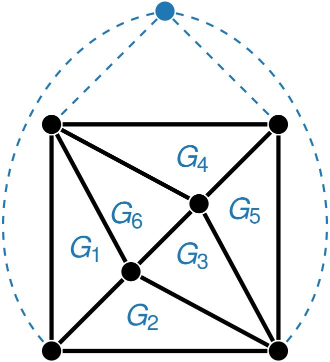
For each tree T with $\Delta(T) \geq 6$, there exists a non-hamiltonian triangulation G, such that T is the decomposition tree of G.

Constructive proof.

Assume $\Delta(T) = 6$.

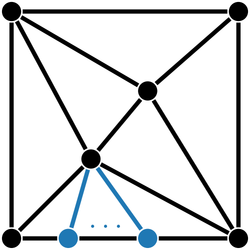


Choose triangulation G_i with decomposition tree T_i $(1 \leq i \leq 6)$



A non-hamiltonian triangulation with T as decomposition tree.

$$\Delta(T) > 6$$



Remaining cases

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

Remaining cases

$$\Delta : \; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; \cdots$$

Not the decomposition tree of non-hamiltonian triangulation

Remaining cases

Δ : 0 1 2 3 | 4 5 | 6 7 $\cdots$

Not the decomposition tree of non-hamiltonian triangulation

Possibly the decomposition tree of non-hamiltonian triangulation

Remaining cases

Δ : 0 1 2 3 | 4 5 | 6 7 $\cdots$

Not the decomposition tree of non-hamiltonian triangulation

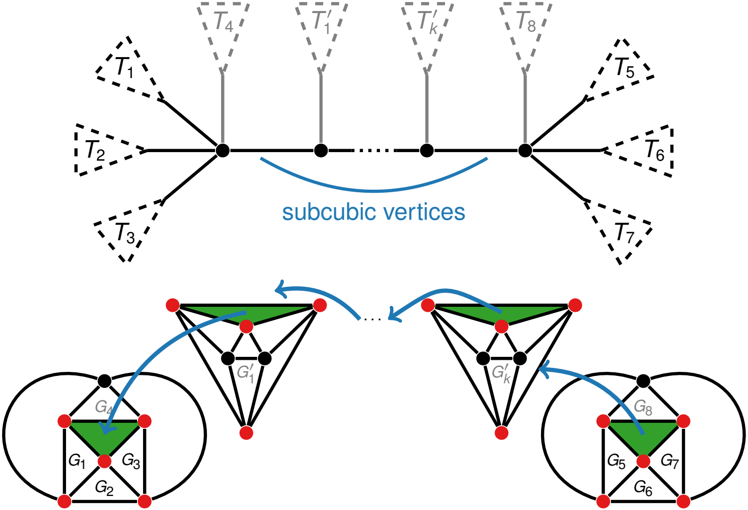
?

Possibly the decomposition tree of non-hamiltonian triangulation

Multiple degrees > 3

Theorem

For each tree T with at least two vertices with degree > 3, there exists a non-hamiltonian triangulation G, such that T is the decomposition tree of G.



red vertices: $5 + k + (5 - 3) = 7 + k$

components: $4 + k + 4 = 8 + k$

Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.

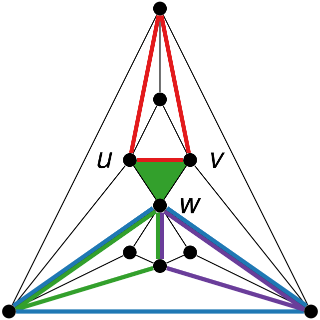
One vertex of degree 4 or 5

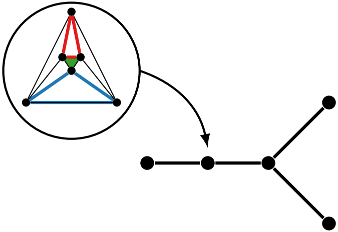
Theorem

Let G be a triangulation with decomposition tree T with only one vertex of degree 4 or 5 and all other vertices of degree at most 3. Then G is 1-tough.

Theorem (Jackson and Yu, 2002)

Let G be a triangulation with decomposition tree T, $\Delta(T) \leq 3$ and uvw a facial triangle of G that is also a facial triangle in a vertex of T with degree at most 2.
Then G has a hamiltonian cycle through uv and vw.





This implies:

Non-hamiltonian triangulations with decomposition trees with one vertex of degree $k \geq 4$ and all others of degree at most 3 exists if and only if...

- non-hamiltonian triangulations with decomposition tree $K_{1,k}$ exist.
- 4-connected triangulations exist with facial triangles $t_1, \ldots, t_k$ so that no hamiltonian cycle C and distinct edges $e_1, \ldots, e_k \in C$ exist such that $e_i \in t_i$.

Also valid for $k \in \{1, 2, 3\}$

Specialised search

Lemma

All triangulations on at most 31 vertices with $K_{1,4}$ as decomposition tree are hamiltonian.

Lemma

All triangulations on at most 27 vertices with $K_{1,5}$ as decomposition tree are hamiltonian.

... and now?

Prove that for each 4-tuple of vertex-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.

... and now?

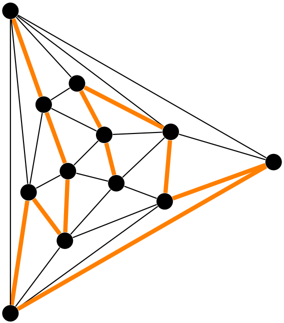
Prove that for each 5-tuple of triangles T_1, T_2, T_3, T_4, T_5 in a 4-connected triangulation there exists a hamiltonian cycle C and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

Find a counterexample.

Hamiltonian path

A hamiltonian path in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $P_{|V|}$.



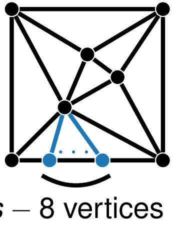
Traceable

A graph $G(V,E)$ is traceable if it contains a hamiltonian path.

Decomposition trees with $\Delta \geq 8$

Theorem

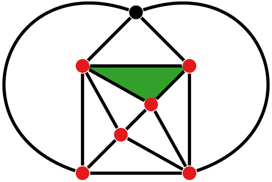
For each tree T with $\Delta(T) \geq 8$, there exists a non-traceable triangulation G, such that T is the decomposition tree of G.



Decomposition trees with $\Delta \in \{6, 7\}$

Theorem

For each tree T with a pair of vertices with degrees k_1 and k_2 with $(k_1, k_2) \in \{(6,4),(6,5),(6,6),(7,4),(7,5),(7,6),(7,7)\}$ and all others of degree at most 3, there exists a non-traceable triangulation G, such that T is the decomposition tree of G.

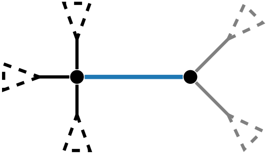


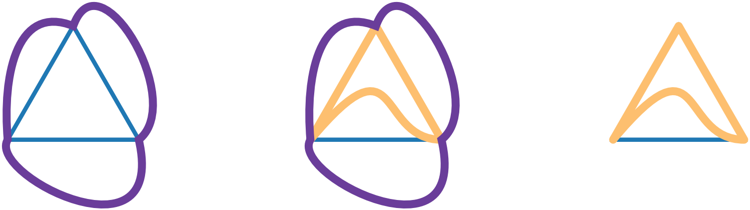
Decomposition trees with $\Delta = 4$

Theorem

Let T be a tree with one vertex of degree 4 and all others of degree at most 3. Then any triangulation which has T as decomposition tree is traceable.

Decomposition trees with $\Delta = 4$





Hamiltonian-connected

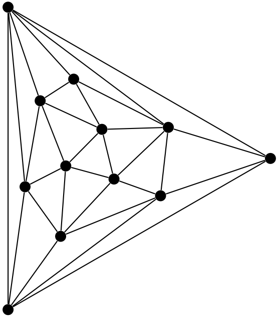
A hamiltonian path connecting x and y is a hamiltonian path P such that x and y have degree 1 in P.

A graph $G(V, E)$ is hamiltonian-connected if for each pair x, y of distinct vertices in V there exists a hamiltonian path connecting x and y.

4-connected triangulations

Theorem (Thomassen, 1983)

Each triangulation without separating triangles is hamiltonian-connected.



3-connected triangulations

Theorem

Let G be a 3-connected triangulation such that there is an edge e which is contained in all separating triangles. Then G is hamiltonian-connected.

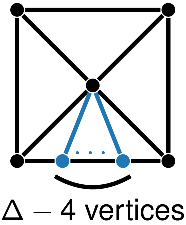
Decomposition tree

Theorem

Let T be a tree with maximum degree 1. Then any triangulation which has T as decomposition tree is hamiltonian-connected.

Theorem

Let T be a tree with maximum degree at least 4. Then T is the decomposition tree of a 3-connected triangulation which is not hamiltonian-connected.



Decomposition tree

Lemma

On up to 21 vertices all triangulations that have a decomposition tree with maximum degree 2 are all hamiltonian-connected.

Lemma

On up to 20 vertices all triangulations that have a decomposition tree with maximum degree 3 are all hamiltonian-connected.

Summary for hamiltonian-connectedness

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

Summary for hamiltonian-connectedness

Δ : 0 1 2 3 4 5 6 7 $\cdots$

Always hamiltonian-connected

Summary for hamiltonian-connectedness

Δ : 0 1 2 3 4 5 6 7 $\cdots$

Always hamiltonian-connected (0, 1)

Possibly not hamiltonian-connected (4, 5, 6, 7, …)

Summary for hamiltonian-connectedness

Δ : 0 1 2 3 4 5 6 7 $\cdots$

- 0, 1: Always hamiltonian-connected
- 2, 3: ?
- 4, 5, 6, 7, …: Possibly not hamiltonian-connected

Overview

Definition

A graph G is k-edge-hamiltonian-connected if for any $X \subset \{x_1x_2 : x_1, x_2 \in V(G), x_1 \neq x_2\}$ such that $1 \leq |X| \leq k$ and X is a forest of paths, $G \cup X$ has a hamiltonian cycle containing all edges in X.

1-edge-hamiltonian-connected is equivalent to hamiltonian-connected.

Definition

A graph G is k-hamiltonian if for any k vertices $v_1, \ldots, v_k$ in G, $G - \{v_1, \ldots, v_k\}$ is hamiltonian.

Table

	Traceable	Hamiltonian	Hamiltonian-connected	2-edge-hamiltonian	1-hamiltonian	2-hamiltonian
0						
1,1						
2,$\ldots$						
3,$\ldots$						
4,($\leq$ 3),$\ldots$						
4,4,$\ldots$						
5,($\leq$ 3),$\ldots$						
5,4,$\ldots$						
5,5,$\ldots$						
6,($\leq$ 3),$\ldots$						
6,4,$\ldots$						
6,5,$\ldots$						
6,6,$\ldots$						
7,($\leq$ 3),$\ldots$						
7,4,$\ldots$						
7,5,$\ldots$						
7,6,$\ldots$						
7,7,$\ldots$						
8,$\ldots$						