Type-0 triangles

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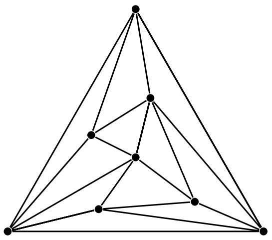
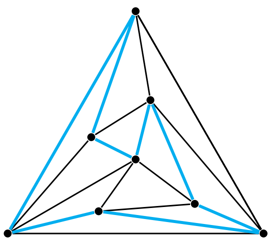
Plane triangulation

A (plane) triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

A hamiltonian cycle C in a graph $G = (V, E)$ is a spanning subgraph of G which is isomorphic to a cycle.



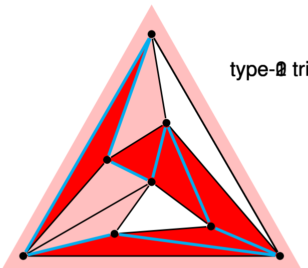
4-connected triangulations

Theorem (Whitney, 1931)

Every 4-connected triangulation is hamiltonian.

Type-i triangle

A type-i triangle ($i \in \{0, 1, 2\}$) in a hamiltonian triangulation G containing a hamiltonian cycle C is a facial triangle of G containing i edges of C.



Type-i triangle

A type-i triangle ($i \in \{0, 1, 2\}$) in a hamiltonian triangulation G containing a hamiltonian cycle C is a facial triangle of G containing i edges of C.

$t_i(G, C) = |\{T, T \text{ is a type-}i \text{ triangle in } G \text{ for } C\}|$

If G and C are clear from the context we just write t_i.

$t_0(G) = \min\{t_i(G, C), C \text{ is hamiltonian cycle in } G\}$

$t_0(t) = \max\{t_i(G), G \text{ is 4-connected triangulation with } t \text{ triangles}\}$

(i, j)-pairs

Let G be a plane triangulation and let C be a hamiltonian cycle in G.

An (i, j)-pair ($i, j \in \{1, 2\}$) is a pair of adjacent triangles consisting of a type-i triangle and a type-j triangle such that the shared edge is contained in C.

2-walk

A 2-walk is a spanning closed walk such that each vertex is visited at most twice.

Theorem (Gao and Richter, 1994)

Every 3-connected plane graph contains a 2-walk.

3-tree

A 3-tree is a spanning tree with maximum degree at most 3.

2-walk vs. 3-tree

Every graph that contains a 2-walk also contains a 3-tree.

3-tree

Theorem (Nakamoto, Oda, and Ota, 2008)

Every 3-connected plane graph on $n \geq 7$ vertices contains a 3-tree with at most $\frac{n-7}{3}$ vertices of degree 3.

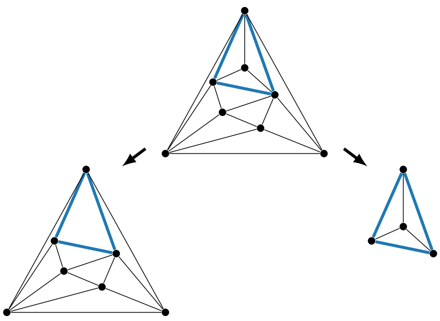
Back to 2-walks

Is there a counterpart of this theorem for 2-walks?

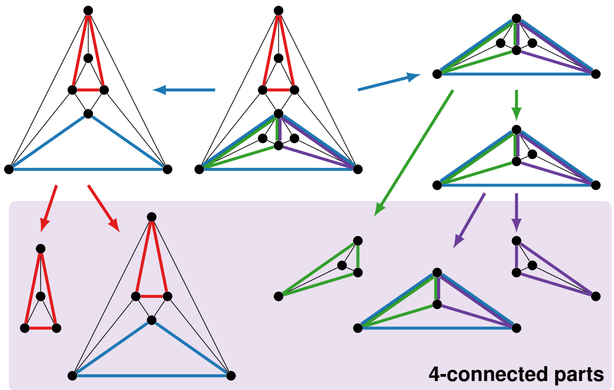
Does each 3-connected plane graph contain a 2-walk such that the number of vertices visited twice is at most $\frac{n}{3}$ + constant?

Does each 3-connected plane triangulation contain such a 2-walk?

Few type-0 triangles



Few type-0 triangles



Few type-0 triangles

Take a hamiltonian cycle in each 4-connected part. If an edge of a separating triangle is contained in such a hamiltonian cycle, then we can detour it to the other side of the hamiltonian cycle ‘without creating a vertex visited twice’.

This is not an exact correspondence, but only an approximation, since specific configurations might still lead to vertices visited twice.

Domination in triangulations

Theorem (Matheson and Tarjan, 1996)

The domination number of any plane triangulation on $n \geq 3$ vertices is at most $\frac{n}{3}$.

Conjecture (Matheson and Tarjan, 1996)

The domination number of any plane triangulation on $n \geq 4$ vertices is at most $\frac{n}{4}$.

Domination in 4-connected triangulations

Theorem (Plummer, Ye, and Zha, 2016)

The domination number of a 4-connected plane triangulation on $n \geq 4$ vertices is at most $\frac{5n}{16}$.

Proof based on hamiltonian cycle with a small number of type-2 triangles.

More precise: if a plane triangulation G contains a hamiltonian cycle with few triangles of type-2 on one side, then G has a 'small' dominating set.

Which type?

Let T be a subcubic tree with V vertices and E edges.
Denote by V_i the number of vertices of degree i.

Counting edges around each vertex gives

$$3V_3 + 2V_2 + V_1 = 2E.$$

Number of edges is one less than number of vertices, so

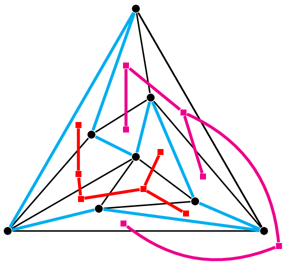
$$V_3 + V_2 + V_1 - 1 = E.$$

Combined this gives

$$V_1 = V_3 + 2.$$

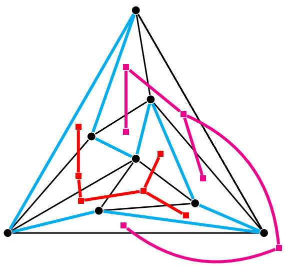
Which type?

Inner dual of either side of a hamiltonian cycle in a plane triangulation is a subcubic tree.



Which type?

Type-i triangles correspond to vertices of degree $3 - i$ in these trees.



Which type?

Using $V_1 = V_3 + 2$, we find

$$t_2 = t_0 + 4.$$

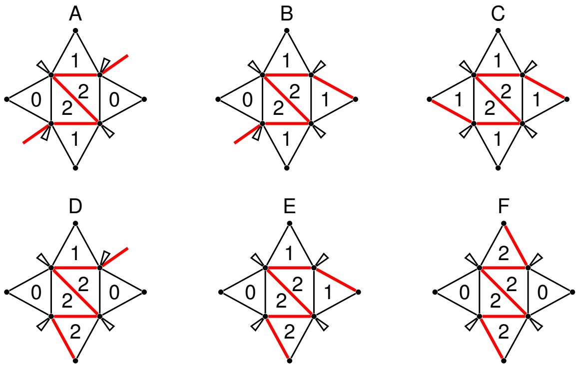
Combining this with $t = t_0 + t_1 + t_2$, we find

$$t_1 = t - 2t_0 - 4.$$

The following are all equivalent:

- finding the minimal value for t_0
- finding the maximal value for t_1
- finding the minimal value for t_2

Neighbourhoods of $(2, 2)$-pairs

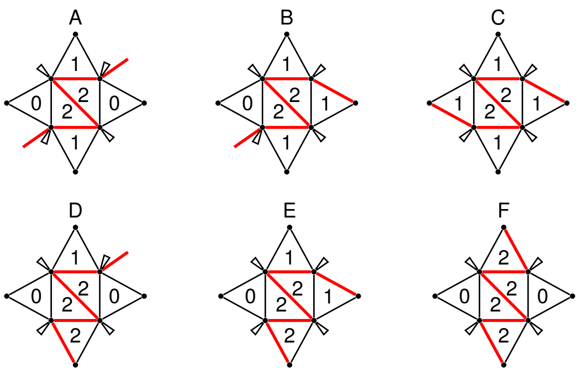


Only certain neighbourhoods

Lemma

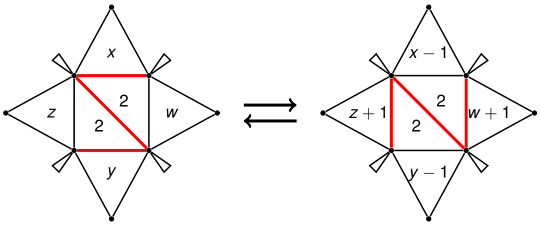
Let G be a 4-connected plane triangulation. Let C be a hamiltonian cycle in G such that C has the smallest number of type-0 triangles among all hamiltonian cycles of G. Then C has no neighbourhood of type D, E, or F.

Neighbourhoods of $(2, 2)$-pairs



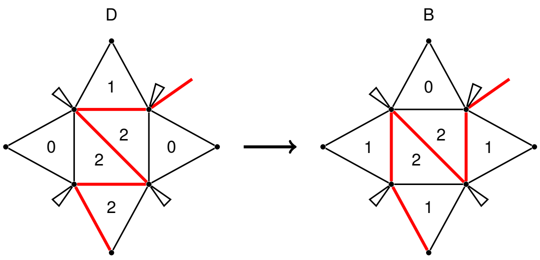
Z-switching

Whenever we have a $(2,2)$-pair, we can perform a Z-switching.



Introduction Motivation Bounds Future Which type? Upper bound Lower bound The gap

Proof of lemma: no D



Brinkmann, Ozeki, Van Cleemput Type-0 triangles

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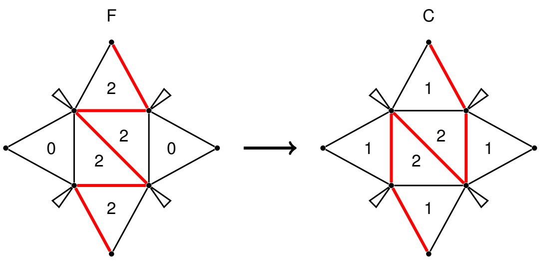
Proof of lemma: no F



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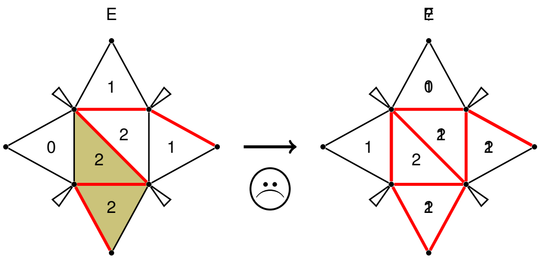
Proof of lemma: E?



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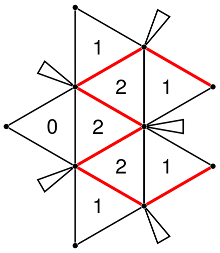
Proof of lemma: EE-neighbourhoods



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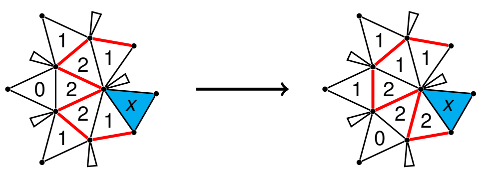
Proof of lemma: moving along the cycle



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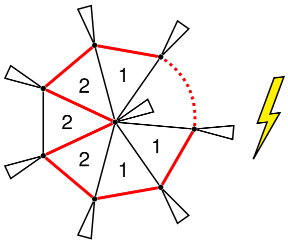
Proof of lemma: only one EE-triangulation



(1,2)-pairs

Corollary

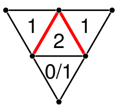
Let G be a 4-connected plane triangulation. Let C be a hamiltonian cycle in G such that C has the smallest number of type-0 triangles among all hamiltonian cycles of G. Then each type-2 triangle for C is contained in at least one (1,2)-pair.

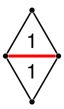
This implies: there are at least as many type-1 triangles as there are type-2 triangles.

At least one non-conformist edge

Lemma

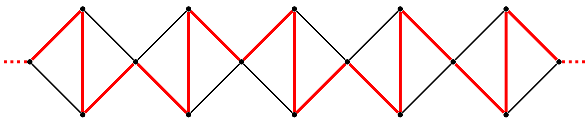
Let G be a 4-connected plane triangulation. Let C be a hamiltonian cycle in G such that C has the smallest number of type-0 triangles among all hamiltonian cycles of G. Then C contains at least one edge that is not incident to a type-2 triangle contained in a (2,2)-pair.



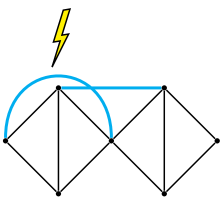


Proof of lemma: a chain of (2,2)-pairs

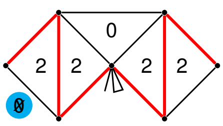
Assume each edge of C is incident to a type-2 triangle contained in a (2,2)-pair.

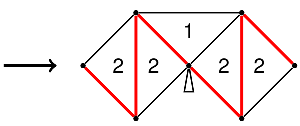


Proof of lemma: position of chord

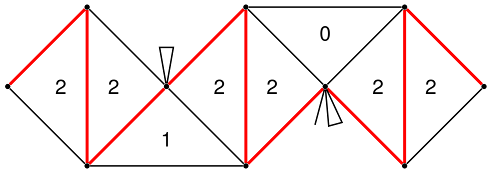


Proof of lemma: hamiltonian cycle near chord triangle

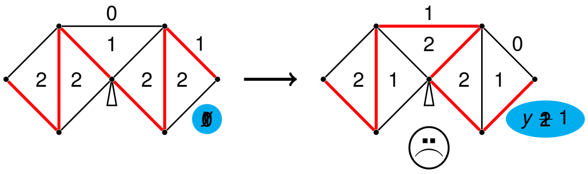




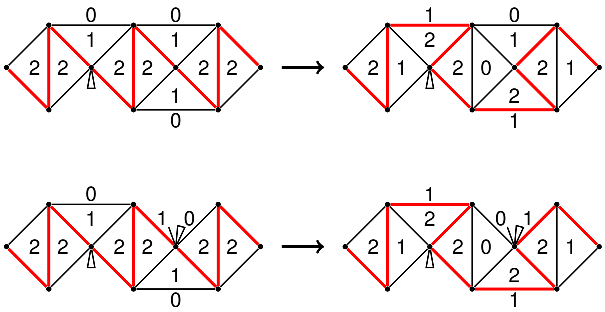
Proof of lemma: existence of type-1 chord triangle



Proof of lemma: rerouting the hamiltonian cycle



Proof of lemma: some more detours



An upper bound for t_0

- Each type-2 triangle is contained in at least one (1,2)-pair.
- There is at least one of the following:
  - a type-2 triangle contained in two (1,2)-pairs, or
  - a (1,1)-pair.

This gives us

$$t_2 \leq t_1 - 1$$

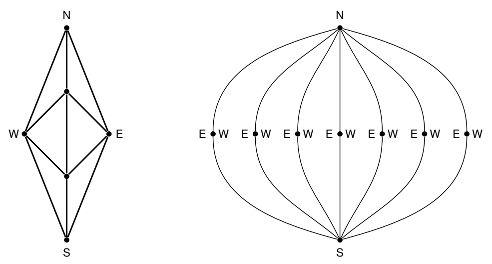
An upper bound for t_0

$$\begin{aligned} t &= t_0 + t_1 + t_2 \\ &= t_0 + t_1 + t_0 + 4 \\ &\geq t_0 + t_0 + 5 + t_0 + 4 \end{aligned}$$

This gives us

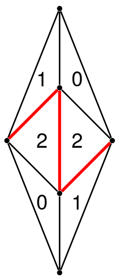
$$t_0 \leq \frac{t}{3} - 3.$$

Lunes



Hamiltonian cycle through lune

If there are k lunes, then at least $k - 8$ lunes contain two type-0 triangles.



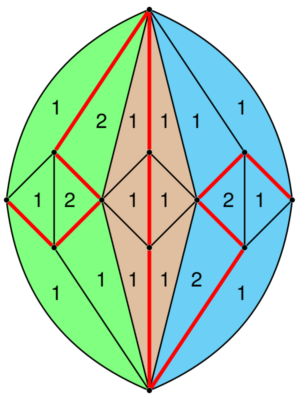
A graph with many type-0 triangles

G_k contains k of these lunes.

$$t_0(G_k) \geq 2k - 16 = \frac{6k - 48}{3} = \frac{t}{3} - 16$$

Lower bound $t_0(G_k)$ is asymptotically equal to upper bound.

Visiting the poles



A lower bound

$$t_0(G_k, C) = 2(k-3) = \frac{6k-18}{3} = \frac{t}{3} - 6$$

C is actually a hamiltonian cycle with the minimum number of type-0 triangles.

$$t_0(G_k) = \frac{t}{3} - 6$$

caagt

The gap

$$\frac{t}{3} - 6 \quad \boxed{\frac{t}{3} - 6} \leq t_0 \leq \frac{t}{3} - 3$$

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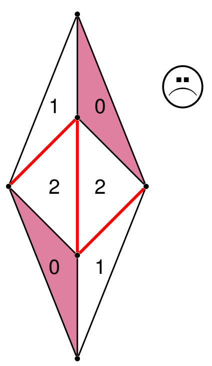
Which side?



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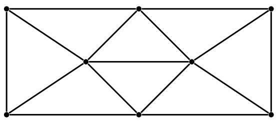
5-connected case

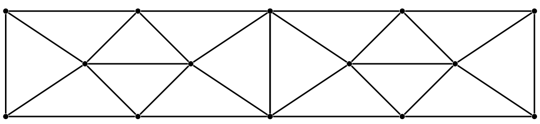
There is a family of 5-connected triangulations with

$$t_0 = \frac{t}{6} - 8$$

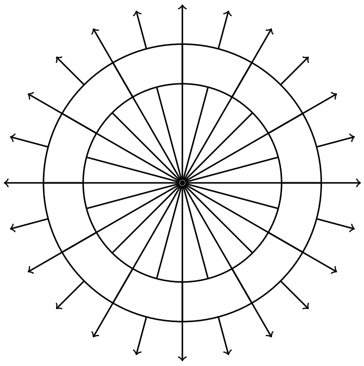
5-connected case



5-connected case

. . .  . . .

5-connected case



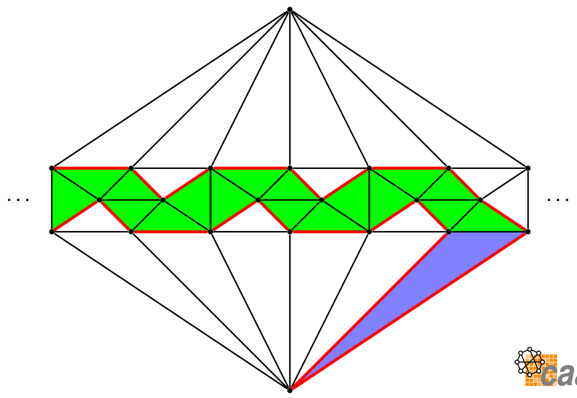
5-connected case

In general at least two type-0 triangles per building block, so this gives

$$t_0 = \frac{t}{6} - 8$$

However, . . .

5-connected case



5-connected case

Conjecture

There exists a family of 5-connected triangulations such that each hamiltonian cycle has a linear number of type-0 triangles on either side.