Generation of Delaney-Dress symbols

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Delaney-Dress symbols?

A Delaney-Dress symbol encodes an equivariant tiling
(i.e. a tiling together with its symmetry group)

The main characters

Definition

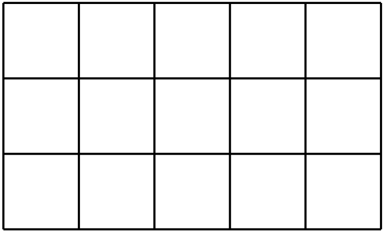
Tiling T = set of tiles $t_1, t_2, \ldots$ with $t_i \subset \mathbb{E}^2$, t_i homeomorph to $\overline{B}(0,1)$, that satisfy the following conditions:

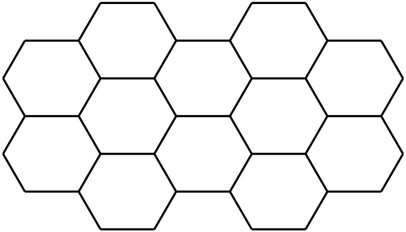
1. $\bigcup_{t \in T} t = \mathbb{E}^2$
2. $\forall t_i, t_j (i \neq j) \in T : t_i^\circ \cap t_j^\circ = \emptyset$ and $t_i \cap t_j$ is empty, point or line.
3. $\forall x \in \mathbb{E}^2$: x has a neighbourhood that only intersects a finite number of tiles.

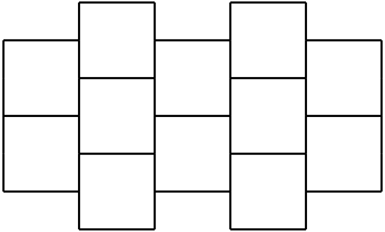
Periodic tiling

symmetry group contains *two independent translations*

caagt



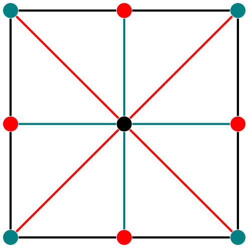




Barycentric subdivision

- For each face: one point
- For each edge: one point
- For each vertex: one point

Incidence determines adjacency

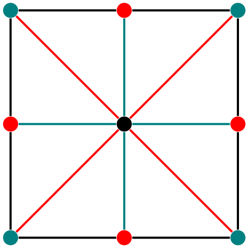


Chamber system

Define $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 | \sigma_i^2 = \mathbb{1} \rangle$

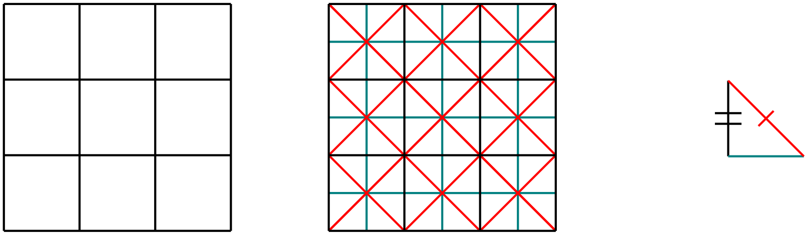
σ_0 : change the green point (vertex).
σ_1 : change the red point (edge).
σ_2 : change the black point (face).

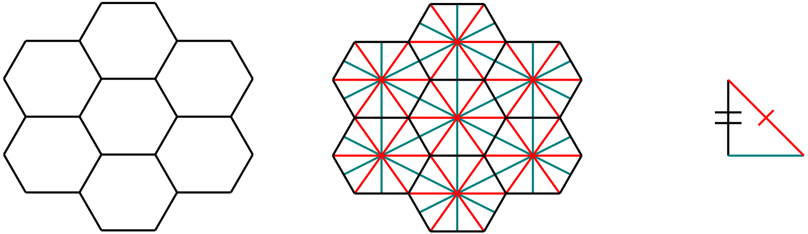
Chamber system $\mathcal{C}$ of T = barycentric subdivision with Σ



Delaney-Dress graph

The Delaney-Dress graph $\mathcal{D}$ of a periodic tiling is the set of *orbits of the chambers* of the chamber system of the tiling under the symmetry group of the tiling.





Observation

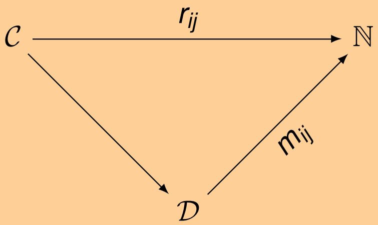
Delaney-Dress graph is not sufficient to distinguish between tilings!

Define functions $r_{ij} : \mathcal{C} \to \mathbb{N}; c \mapsto r_{ij}(c)$ with $r_{ij}(c)$ the smallest number for which $c(\sigma_i\sigma_j)^{r_{ij}(c)} = c$.

Observation

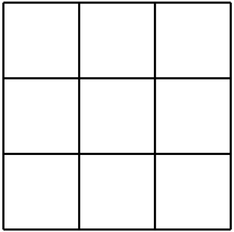
r_{02} is a constant function with value 2.
$r_{01}(c)$ is the size of the face of T that belongs to c.
$r_{12}(c)$ is the number of faces that meet in the vertex that belongs to c.

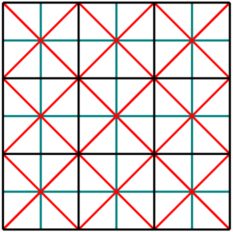
Define functions $m_{ij} : \mathcal{D} \to \mathbb{N}; d \mapsto m_{ij}(c)$ in such a manner that the following diagram is commutative:

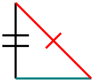


Delaney-Dress symbol

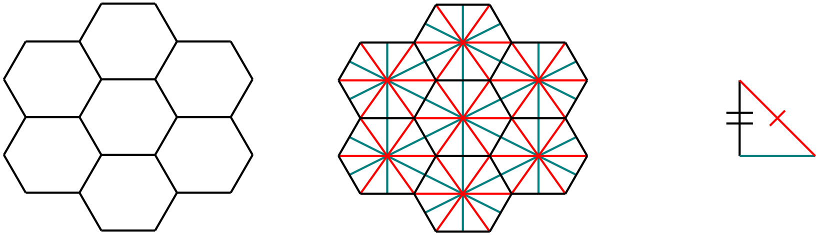
The Delaney-Dress symbol of a periodic tiling is $(\mathcal{D}; m_{01}, m_{12})$.







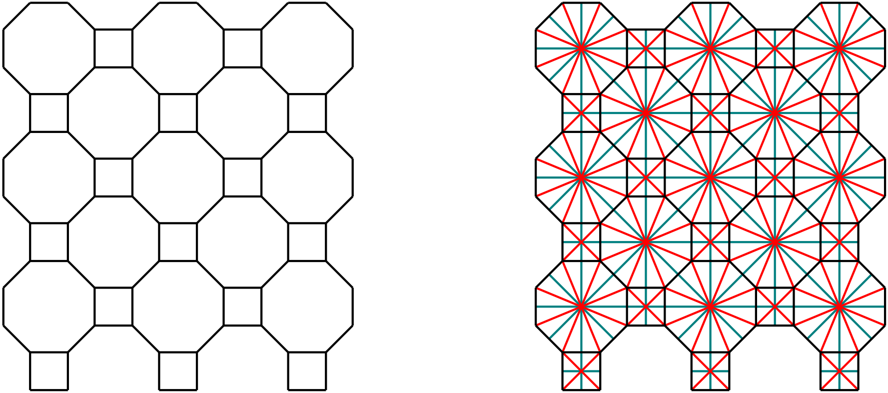
$m_{01}(c) = 4$
$m_{12}(c) = 4$

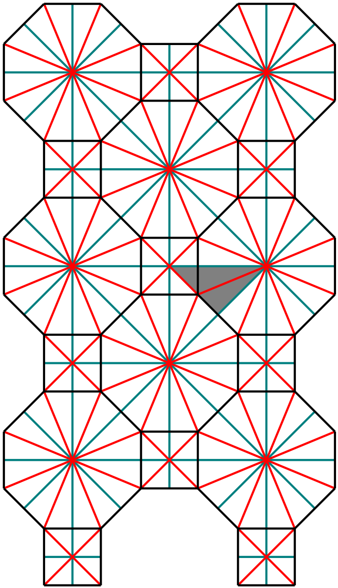


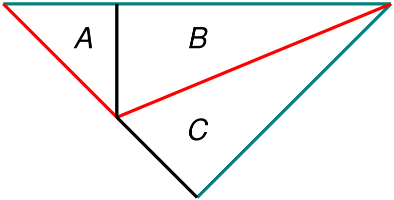
$m_{01}(c) = 6$
$m_{12}(c) = 3$

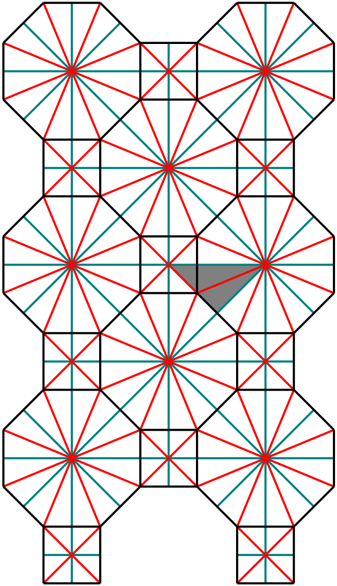
Theorem (Dress, 1985)

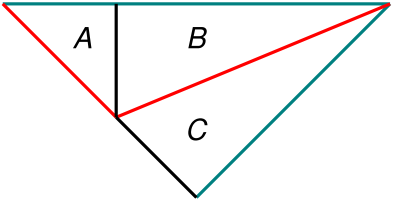
Two tilings are equivariantly equivalent iff their respective Delaney-Dress symbols are isomorphic.



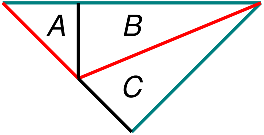


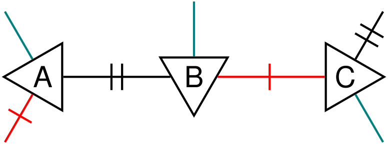

A
B
C





	m_{01}	m_{12}
A	4	3
B	8	3
C	8	3





	m_{01}	m_{12}
A	4	3
B	8	3
C	8	3

Theorem (Dress et al., 1980s-1990s)

$(\mathcal{D}; m_{01}, m_{12})$ is the Delaney-Dress symbol of a periodic tiling iff

1. *$\mathcal{D}$ is finite*
2. *Σ works transitively on $\mathcal{D}$*
3. *m_{01} is constant on $\langle \sigma_0, \sigma_1 \rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_0\sigma_1)^{m_{01}(d)} = d$*
4. *m_{12} is constant on $\langle \sigma_1, \sigma_2 \rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_1\sigma_2)^{m_{12}(d)} = d$*
5. *$\forall d \in \mathcal{D} : d(\sigma_0\sigma_2)^2 = d$*
6. *Curvature condition*

Curvature condition

$$K(\mathcal{D}) = \sum_{d\in\mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2}\right)$$

- $K(\mathcal{D}) < 0 \rightarrow$ hyperbolic plane
- $K(\mathcal{D}) = 0 \rightarrow$ euclidean plane
- $K(\mathcal{D}) > 0 \rightarrow$ sphere iff $\frac{4}{K(\mathcal{D})} \in \mathbb{N}$

Generation of Delaney-Dress symbols

Motivation

- Can we generate all cubic pregraphs?
  (i.e. multigraphs with loops and semi-edges)
- Can we filter out the 3-edge-colourable pregraphs?
- Can we filter out the underlying graphs of Delaney-Dress graphs?

Why this wasn't the best modus operandi

n	colourable	Delaney-Dress	ratio
1	1	1	100.00 %
2	3	3	100.00 %
3	3	2	66.67%
4	11	9	81.82%
5	17	7	41.18%
6	59	29	49.15%
7	134	27	20.15%
8	462	105	22.73%
9	1 332	118	8.86%
10	4 774	392	8.21%
11	16 029	546	3.41%
12	60 562	1 722	2.84%
13	225 117	2 701	1.20%
14	898 619	7 953	0.89%
15	3 598 323	13 966	0.39%
16	15 128 797	40 035	0.26%
17	64 261 497	75 341	0.12%
18	283 239 174	210 763	0.07%
19	1 264 577 606	420 422	0.03%
20	5 817 868 002	1 162 192	0.02%

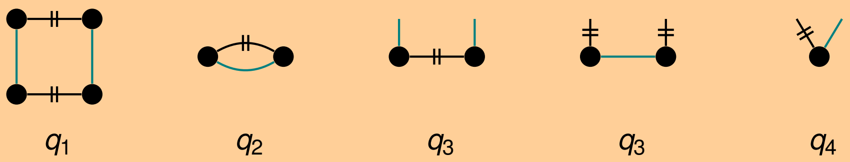
The structure of Delaney-Dress graphs

From the theorem...

1. $\mathcal{D}$ is finite
2. Σ works transitively on $\mathcal{D}$
5. $\forall d \in \mathcal{D} : d(\sigma_0\sigma_2)^2 = d$

Translated:

Finite, connected, 3-edge-coloured pregraphs where each 02-component is isomorphic to one of



C_4^q-marked pregraphs

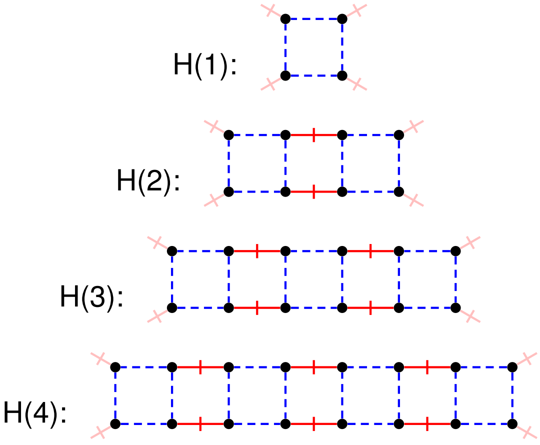
Pregraphs together with a 2-factor for which each component is a quotient of C_4.

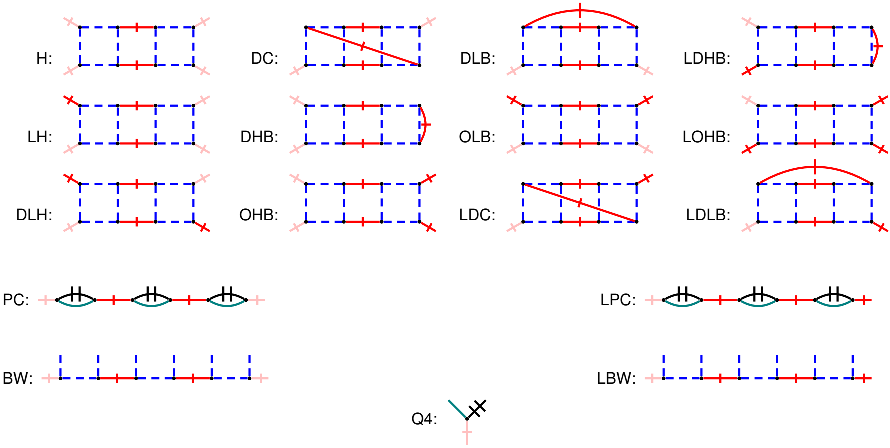
For each C_4^q-marked pregraph, there exists a unique partition of the graph into subgraphs of some specific types

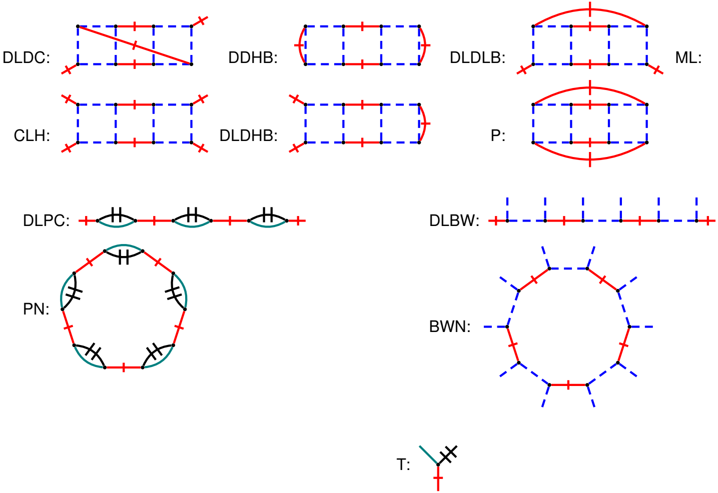
Block partition

- maximal ladders containing only marked quotients of type q_1
- maximal subgraphs induced by marked quotients of type q_2
- maximal subgraphs induced by marked quotients of type q_3
- marked quotients of type q_4

Example of a parameterized block




H:
DC:
DLB:
LDHB:
LH:
DHB:
OLB:
LOHB:
DLH:
OHB:
LDC:
LDLB:
PC:
LPC:
BW:
LBW:
Q4:


DLDC:
DDHB:
DLDLB:
ML:
CLH:
DLDHB:
P:
DLPC:
DLBW:
PN:
BWN:
T:

Generating the C_4^q-marked pregraphs

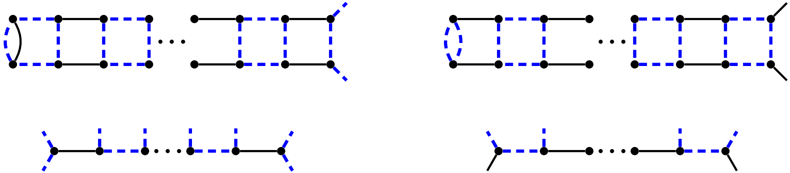
1. Generate lists of blocks
2. Connect blocks in list

C_4^q-markable pregraphs

- Underlying graphs of C_4^q-marked pregraphs
- Generated by `pregraphs`
- Easy to derive C_4^q-markable pregraphs
- Most C_4^q-markable pregraphs have a unique C_4^q-2-factor

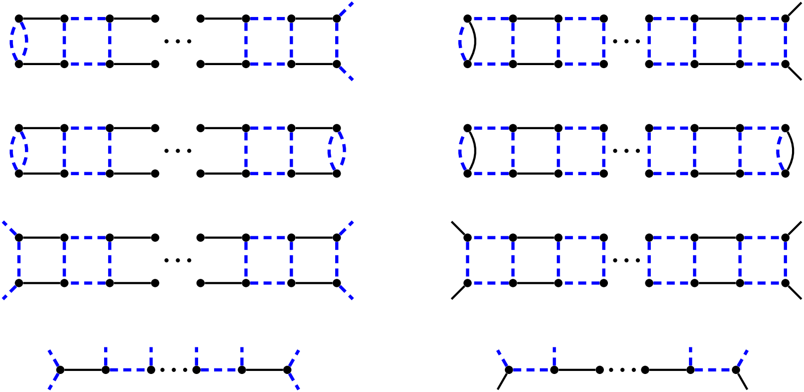
C_4^q-markable pregraphs with n vertices

- n odd: each C_4^q-markable pregraph has a unique C_4^q-2-factor
- $n \bmod 4 \equiv 2$



C_4^q-markable pregraphs with n vertices

- $n \bmod 4 \equiv 0$

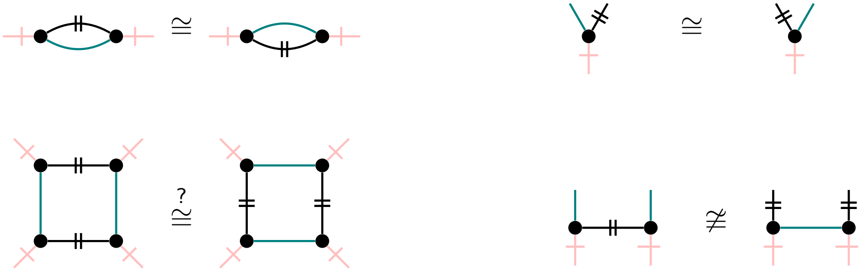


Generating the Delaney-Dress graphs

1. Generate lists of blocks
2. Connect blocks in list
3. Assign missing colours

(1, 2) → C_4^q-marked pregraphs

The last colours



Partially coloured Delaney-Dress graphs

- uncoloured quotients are of type q_1 and q_3
- $U =$ set of uncoloured quotients
- colour assignment can be represented by a bit vector of length $|U|$
  - number uncoloured quotients
  - choose a matching in each quotient of type q_1
  - 0 if edges in matching in quotient of type q_1 receive colour 0
    0 if the semi-edges in quotient of type q_3 receive colour 0
- efficiently check which colour assignments are isomorphic

Generating the Delaney-Dress symbols

1. Generate lists of blocks
2. Connect blocks in list
3. Assign missing colours
4. Determine functions m_{01} and m_{12}

(1) Generate lists of blocks and (2) Connect blocks in list → C_4^q-marked pregraphs

(3) Assign missing colours → Delaney-Dress graphs

From the theorem...

3. m_{01} is constant on $\langle\sigma_0,\sigma_1\rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_0\sigma_1)^{m_{01}(d)} = d$
4. m_{12} is constant on $\langle\sigma_1,\sigma_2\rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_1\sigma_2)^{m_{12}(d)} = d$
6. $\sum_{d\in\mathcal{D}}(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2}) = 0$

$F =$ set of 0,1-components (faces)
$V =$ set of 1,2-components (vertices)

m_{01}, resp. m_{12} is constant on elements of F, resp. V.

$m_F : F \to \mathbb{N}; f \mapsto m_F(f) = m_{01}(d)$ with $d \in f$
$m_V : V \to \mathbb{N}; v \mapsto m_V(v) = m_{12}(d)$ with $d \in v$

$$0 = \sum_{f \in F} \frac{|f|}{m_F(f)} + \sum_{v \in V} \frac{|v|}{m_V(v)} - \frac{|\mathcal{D}|}{2}$$

Crystallographic restriction theorem

The rotational symmetries in the Euclidean plane are 2-fold, 3-fold, 4-fold or 6-fold.

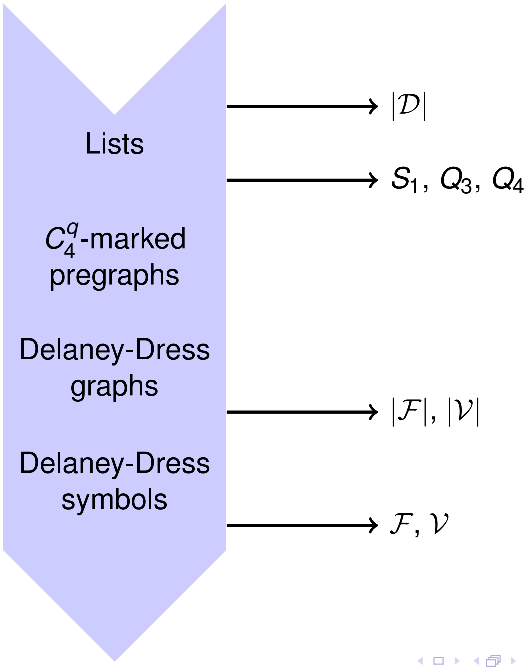
For any i, j-component O, $o \in O$:

$$m_{ij}(o) \in \left\{6|O|, 4|O|, 3|O|, 2|O|, |O|, \frac{3|O|}{2}, \frac{|O|}{2}\right\}$$

A step back: limiting the generated class

Notation	Description
$M_{\lvert\mathcal{F}\rvert}$	Maximum number of face orbits allowed
$m_{\lvert\mathcal{F}\rvert}$	Minimum number of face orbits required
$M_{\lvert\mathcal{V}\rvert}$	Maximum number of vertex orbits allowed
$m_{\lvert\mathcal{V}\rvert}$	Minimum number of vertex orbits required
$M_{\mathcal{F}}$	Maximum value of m_{01} allowed
$m_{\mathcal{F}}$	Minimum value of m_{01} required
$M_{\mathcal{V}}$	Maximum value of m_{12} allowed
$m_{\mathcal{V}}$	Minimum value of m_{12} required
$\mathcal{R}_{\mathcal{F}}$	Multiset of required face sizes
$\mathcal{R}_{\mathcal{V}}$	Multiset of required vertex degrees
$\mathcal{U}_{\mathcal{F}}$	Set of forbidden (unwanted) face sizes
$\mathcal{U}_{\mathcal{V}}$	Set of forbidden (unwanted) vertex degrees

Property	Description
$\|\mathcal{D}\|$	The number of flags in the Delaney-Dress graph
$\mathcal{F}$	The multiset of sizes of a face in the face orbits
$\mathcal{V}$	The multiset of degrees of a vertices in the vertex orbits
$\|\mathcal{F}\|$	The number of face orbits
$\|\mathcal{V}\|$	The number of vertex orbits
S_1	The number of semi-edges with colour 1
Q_3	The number of q_3 components
Q_4	The number of q_4 components


Lists
C_4^q-marked pregraphs
Delaney-Dress graphs
Delaney-Dress symbols
$|\mathcal{D}|$
S_1, Q_3, Q_4
$|\mathcal{F}|$, $|\mathcal{V}|$
$\mathcal{F}$, $\mathcal{V}$

Refining the restrictions

$$\sum_{d\in\mathcal{D}}\left(\frac{1}{M_{\mathcal{F}}}+\frac{1}{M_{\mathcal{V}}}-\frac{1}{2}\right)\leq\sum_{d\in\mathcal{D}}\left(\frac{1}{m_{01}(d)}+\frac{1}{m_{12}(d)}-\frac{1}{2}\right)\leq\sum_{d\in\mathcal{D}}\left(\frac{1}{m_{\mathcal{F}}}+\frac{1}{m_{\mathcal{V}}}-\frac{1}{2}\right),$$

$$0\leq 2m_{\mathcal{V}}+2m_{\mathcal{F}}-m_{\mathcal{V}}m_{\mathcal{F}}$$

$$0\geq 2M_{\mathcal{V}}+2M_{\mathcal{F}}-M_{\mathcal{V}}M_{\mathcal{F}}$$

$$0 \leq 2m_{\mathcal{V}} + 2m_{\mathcal{F}} - m_{\mathcal{V}}m_{\mathcal{F}}$$

	3	4	5	6	7
3	3	2	1	0	-1
4	2	0	-2	-4	-6
5	1	-2	-5	-8	-11
6	0	-4	-8	-12	-16
7	-1	-6	-11	-16	-21

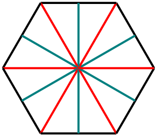
$$0 \geq 2M_{\mathcal{V}} + 2M_{\mathcal{F}} - M_{\mathcal{V}}M_{\mathcal{F}}$$

	3	4	5	6	7
3	3	2	1	0	-1
4	2	0	-2	-4	-6
5	1	-2	-5	-8	-11
6	0	-4	-8	-12	-16
7	-1	-6	-11	-16	-21

Possible orders of Delaney-Dress graphs

Generation of Delaney-Dress graphs with given order

Try to limit the number of possible orders



$$K = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$0 \leq 2|\mathcal{F}| + \frac{|\mathcal{D}|}{m_{\mathcal{V}}} - \frac{|\mathcal{D}|}{2}$$

$$0 \leq 2M_{|\mathcal{F}|} + \frac{|\mathcal{D}|}{m_{\mathcal{V}}} - \frac{|\mathcal{D}|}{2}$$

$$|\mathcal{D}| \leq \left(\frac{4m_{\mathcal{V}}}{m_{\mathcal{V}} - 2} \right) M_{|\mathcal{F}|}$$

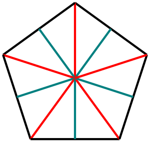
Similar:

$$|\mathcal{D}| \leq \left(\frac{4m_{\mathcal{F}}}{m_{\mathcal{F}} - 2} \right) M_{|\mathcal{V}|}$$

$$K = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$0 \leq 2|\mathcal{F}| + 2|\mathcal{V}| - \frac{|\mathcal{D}|}{2}$$

$$|\mathcal{D}| \leq 4\left(|\mathcal{V}| + |\mathcal{F}|\right) \leq 4\left(M_{|\mathcal{V}|} + M_{|\mathcal{F}|}\right)$$

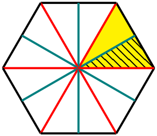


$$|\mathcal{D}| \leq \sum_{f \in \mathcal{F}} 2f$$

$$|\mathcal{D}| \leq 2\left(\sum_{f \in \mathcal{R}_\mathcal{F}} f\right) + 2(M_{|\mathcal{F}|} - |\mathcal{R}_\mathcal{F}|)M_\mathcal{F}$$

Similiar:

$$|\mathcal{D}| \leq 2\left(\sum_{v \in \mathcal{R}_\mathcal{V}} v\right) + 2(M_{|\mathcal{V}|} - |\mathcal{R}_\mathcal{V}|)M_\mathcal{V}$$



$$\mathcal{C} : \mathbb{N} \to \mathbb{N}; n \mapsto \mathcal{C}(n) = \begin{cases} \frac{n}{6} & n \bmod 6 = 0 \\ \frac{n}{4} & n \bmod 4 = 0 \text{ and } n \bmod 3 \neq 0 \\ \frac{n}{3} & n \bmod 3 = 0 \text{ and } n \bmod 2 \neq 0 \\ \frac{n}{2} & n \bmod 2 = 0 \text{ and } n \bmod 3 \neq 0 \text{ and } n \bmod 4 \neq 0 \\ n & \text{all other cases} \end{cases}$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{F}} \mathcal{C}(f)$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{R}_{\mathcal{F}}} \mathcal{C}(f) + (m_{|\mathcal{F}|} - |\mathcal{R}_{\mathcal{F}}|) \min\{\mathcal{C}(n) | m_{\mathcal{F}} \leq n \leq M_{\mathcal{F}} \wedge n \notin \mathcal{U}_{\mathcal{F}}\}$$

Similar:

$$|\mathcal{D}| \geq \sum_{v \in \mathcal{R}_\mathcal{V}} \mathcal{C}(v) + (m_{|\mathcal{V}|} - |\mathcal{R}_\mathcal{V}|) \min\{\mathcal{C}(n) | m_\mathcal{V} \leq n \leq M_\mathcal{V} \wedge n \notin \mathcal{U}_\mathcal{V}\}$$

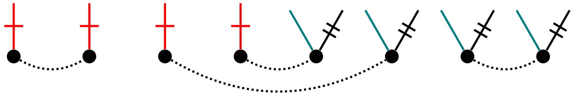
Example

Restriction	Calculated	Actual
$M_{\|\mathcal{F}\|} = 1$	$\|\mathcal{D}\| \in [1, 12]$	$\|\mathcal{D}\| \in [1, 12]$
	$\|\mathcal{F}\| = 1$	$\|\mathcal{F}\| = 1$
	$\|\mathcal{V}\| \in [1, 12]$	$\|\mathcal{V}\| \in [1, 4]$
	$\mathcal{F} \subset [3, 144]$	$\mathcal{F} \subset [3, 6]$
	$\mathcal{V} \subset [3, 144]$	$\mathcal{V} \subset [3, 12]$

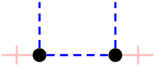
Rejecting lists of blocks

n	lists	time	C_4^q-marked	time
16	6 776	0.0s	40 039	1.5s
17	5 171	0.0s	75 341	2.2s
18	16 557	0.0s	210 765	8.0s
19	12 321	0.0s	420 422	14.0s
20	40 622	0.1s	1 162 196	46.6s
21	29 843	0.1s	2 419 060	86.7s
22	93 166	0.2s	6 626 610	273.7s
23	67 345	0.2s	14 292 180	551.9s
24	213 822	0.5s	38 958 571	1 704.0s
25	153 388	0.5s	86 488 183	3 586.2s
26	467 050	1.2s	235 004 260	10 714.7s
27	331 411	1.2s	534 796 010	23 619.7s
28	1 018 009	3.0s	1 450 990 715	69 251.9s
29	719 250	2.9s	3 373 088 492	157 167.0s
30	2 136 996	6.8s	9 147 869 420	455 606.1s

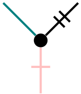
Limit the length of chains of digons



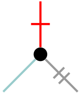
$$|\mathcal{V}| \geq \left\lceil \frac{S_1 + Q_4}{2} \right\rceil \qquad |\mathcal{F}| \geq \left\lceil \frac{S_1 + Q_4}{2} \right\rceil$$



$$2 \times \frac{1}{2}\ 0,1\text{-component or } 2 \times \frac{1}{2}\ 1,2\text{-component}$$



$$\frac{1}{2}\ 0,1\text{-component and } \frac{1}{2}\ 1,2\text{-component}$$



$$\frac{1}{2}\ 0,1\text{-component and } \frac{1}{2}\ 1,2\text{-component}$$

$$2\left\lceil \frac{S_1 + Q_4}{2} \right\rceil + Q_3 \leq |\mathcal{V}| + |\mathcal{F}|$$

Rejecting Delaney-Dress graphs

Number of components and their sizes are known

Testing

C_4^q-marked pregraphs

→ Compared to `pregraphs`

Delaney-Dress graphs

→ Independently verified by Alen Orbanić up to 10 vertices

Delaney-Dress symbols (all)

→ Not yet tested

Testing

Delaney-Dress symbols (restricted)
$\rightarrow$ Compared to known enumeration of tilings

- 93 equivariant tile-transitive tilings of the Euclidean plane
- 1270 equivariant tile-2-transitive tilings of the Euclidean plane
- 30 equivariant edge-transitive tilings of the Euclidean plane
- 37 equivariant minimal, non-transitive tilings of the Euclidean plane

Results

n	lists	C_4^q-marked	C_4^q-markable	time `ddgraphs`	time `pregraphs`
1	1	1	1	0.0s	0.0s
2	5	5	3	0.0s	0.0s
3	2	2	2	0.0s	0.0s
4	13	13	9	0.0s	0.0s
5	7	7	7	0.0s	0.0s
6	31	31	29	0.0s	0.0s
7	25	27	27	0.0s	0.0s
8	103	109	105	0.0s	0.0s
9	86	118	118	0.0s	0.0s
10	311	394	392	0.0s	0.1s
11	260	546	546	0.0s	0.3s
12	938	1 726	1 722	0.1s	1.3s
13	763	2 701	2 701	0.1s	5.2s
14	2 521	7 955	7 953	0.3s	22.0s
15	1 968	13 966	13 966	0.4s	94.8s
16	6 776	40 039	40 035	1.5s	420.5s
17	5 171	75 341	75 341	2.2s	1 903.5s
18	16 557	210 765	210 763	8.0s	8 850.1s
19	12 321	420 422	420 422	14.0s	41 812.1s
20	40 622	1 162 196	1 162 192	46.6s	201 745.4s

n	lists	time `ddgraphs`	C_4^q-marked	C_4^q-markable	time `ddgraphs`
21	29 843	0.1s	2 419 060	2 419 060	86.7s
22	93 166	0.2s	6 626 610	6 626 608	273.7s
23	67 345	0.2s	14 292 180	14 292 180	551.9s
24	213 822	0.5s	38 958 571	38 958 567	1 704.0s
25	153 388	0.5s	86 488 183	86 488 183	3 586.2s
26	467 050	1.2s	235 004 260	235 004 258	10 714.7s
27	331 411	1.2s	534 796 010	534 796 010	23 619.7s
28	1 018 009	3.0s	1 450 990 715	1 450 990 711	69 251.9s
29	719 250	2.9s	3 373 088 492	3 373 088 492	157 167.0s
30	2 136 996	6.8s	9 147 869 420	9 147 869 418	455 606.1s

n	Delaney-Dress graphs	time	rate
16	433 030	1.6s	270 643.75/s
17	931 729	2.5s	372 691.60/s
18	3 196 841	9.1s	351 301.21/s
19	7 258 011	16.3s	445 276.75/s
20	24 630 262	55.0s	447 822.95/s
21	58 309 071	105.9s	550 605.01/s
22	196 266 434	345.5s	568 064.93/s
23	481 330 615	722.2s	666 478.28/s
24	1 610 942 856	2 329.2s	691 629.25/s
25	4 071 117 829	5 184.9s	785 187.34/s
26	13 569 014 653	16 422.1s	826 265.50/s
27	35 202 390 477	38 273.5s	919 758.85/s
28	116 994 675 348	121 796.3s	960 576.60/s
29	310 624 700 725	295 889.0s	1 049 801.45/s
30	1 030 455 432 427	949 823.0s	1 084 892.06/s

n	Delaney-Dress symbols	time
1	3	0.0s
2	15	0.0s
3	8	0.0s
4	37	0.0s
5	15	0.0s
6	86	0.0s
7	64	0.0s
8	217	0.2s
9	185	0.5s
10	527	3.8s
11	506	13.0s
12	1 597	95.1s
13	1 575	360.4s
14	4 227	2 531.5s
15	4 532	10 383.8s
16	12 078	70 331.9s
17	13 105	304 083.2s
18	34 250	1 994 897.8s

Future work

n	used	unused	ratio used
1	1	0	100.00%
2	7	0	100.00%
3	3	0	100.00%
4	20	2	90.91%
5	7	6	53.85%
6	35	35	50.00%
7	18	49	26.87%
8	90	225	28.57%
9	63	330	16.03%
10	163	1414	10.34%
11	161	2354	6.40%
12	452	9028	4.77%
13	436	16769	2.53%
14	1089	60505	1.77%
15	1323	122630	1.07%
16	2997	430033	0.69%
17	3747	927982	0.40%
18	8048	3188793	0.25%

Future work

Restriction	Calculated	Actual
	$\lvert\mathcal{D}\rvert \in [1, 12]$	$\lvert\mathcal{D}\rvert \in [1, 12]$
	$\lvert\mathcal{F}\rvert = 1$	$\lvert\mathcal{F}\rvert = 1$
$M_{\lvert\mathcal{F}\rvert} = 1$	$\lvert\mathcal{V}\rvert \in [1, 12]$	$\lvert\mathcal{V}\rvert \in [1, 4]$
	$\mathcal{F} \subset [3, 144]$	$\mathcal{F} \subset [3, 6]$
	$\mathcal{V} \subset [3, 144]$	$\mathcal{V} \subset [3, 12]$

Future work

Restriction	Calculated	Actual
	$\lvert\mathcal{D}\rvert \in [2, 24]$	$\lvert\mathcal{D}\rvert \in [2, 24]$
$m_{\lvert\mathcal{F}\rvert} = 2$	$\lvert\mathcal{F}\rvert = 2$	$\lvert\mathcal{F}\rvert = 2$
$M_{\lvert\mathcal{F}\rvert} = 2$	$\lvert\mathcal{V}\rvert \in [1, 24]$	$\lvert\mathcal{V}\rvert \in [1, 6]$
	$\mathcal{F} \subset [3, 276]$	$\mathcal{F} \subset [3, 24]$
	$\mathcal{V} \subset [3, 288]$	$\mathcal{V} \subset [3, 24]$

Thank you for your attention.