Generation of generalized 3-regular graphs

N. Van Cleemput[1] G. Brinkmann[1] T. Pisanski[2]

[1]Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics and Computer Science
Ghent University

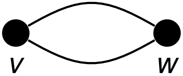
[2]Department of Theoretical Computer Science
University of Ljubljana

Which structures will be generated?

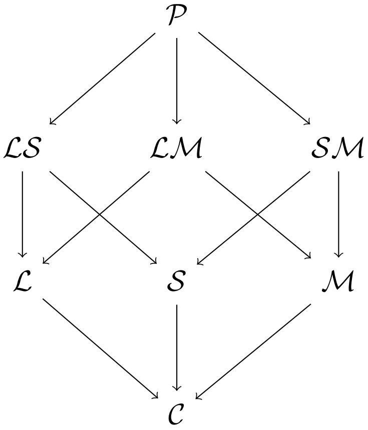
3-regular variety of

- simple graphs
- multigraphs
- graphs with loops
- graphs with semi-edges
- any combination of these

Which structures will be generated?

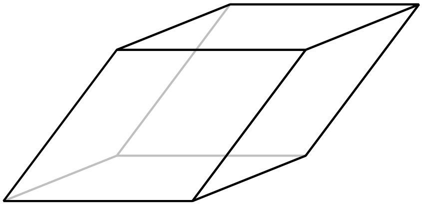
Name	Type	Counts as
Loop		2
Multi-edge		2
Semi-edge		1

Which structures will be generated?



Motivation

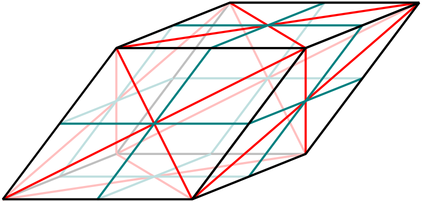
- Study of maps
  - flag graphs of maps / hypermaps
  - symmetry type graphs / Delaney-Dress graphs
  - arc graphs of oriented maps
- Voltage graphs

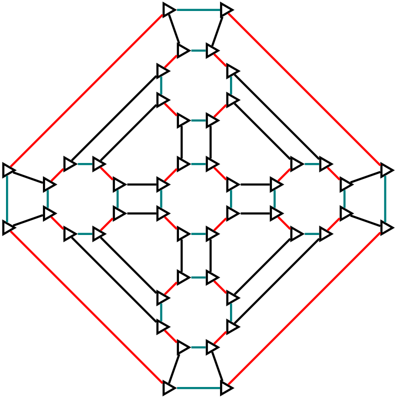


Rhombohedron

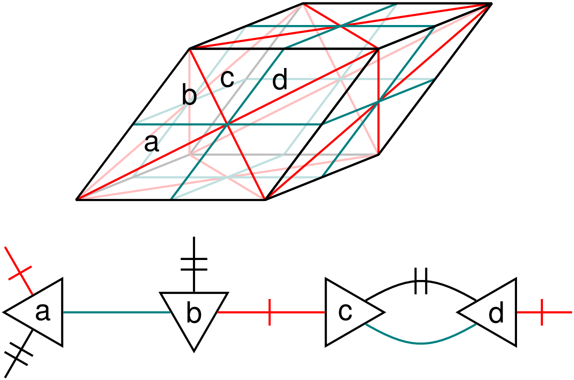
all 6 faces are congruent rhombi
has D_{3d} symmetry
(Trigonal trapezohedron)

Motivation - Delaney-Dress graph

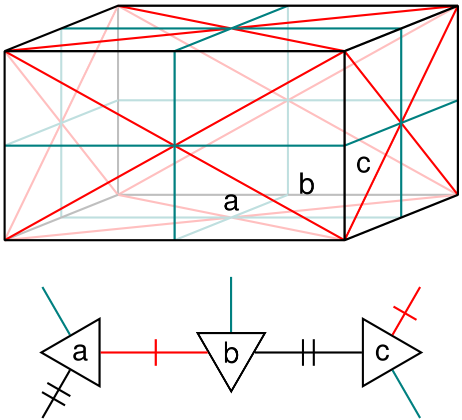




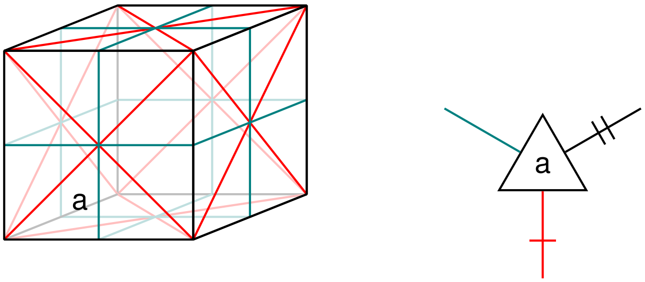
Motivation - Delaney-Dress graph



Motivation - Delaney-Dress graph

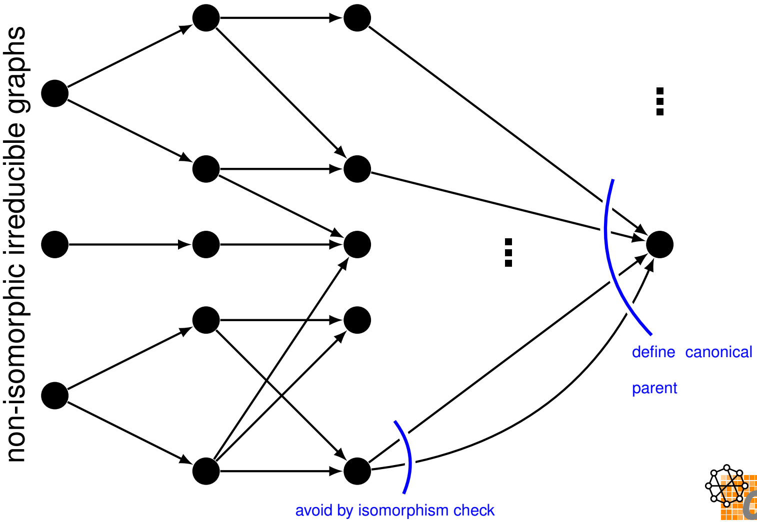


Motivation - Delaney-Dress graph



Which technique will be used?

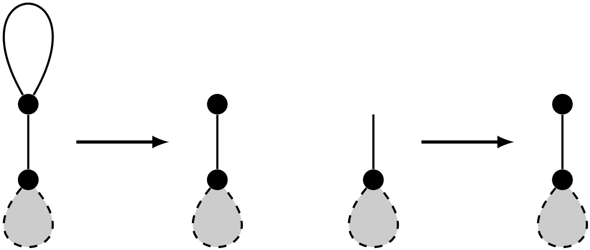
Canonical construction path



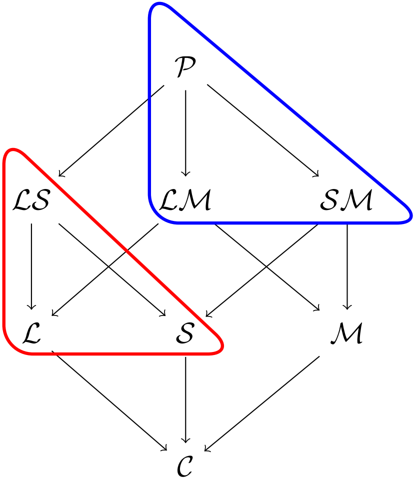
Translation to multigraphs

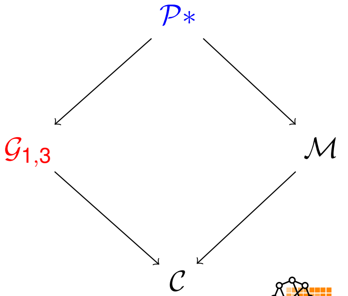
Pregraph primitives

Translate cubic pregraphs to multigraphs with degrees 1 and 3.
Notation: $*(G)$ is the primitive of G.

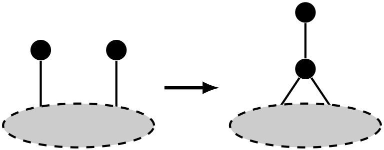


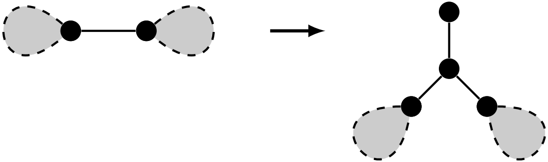
Translation to multigraphs



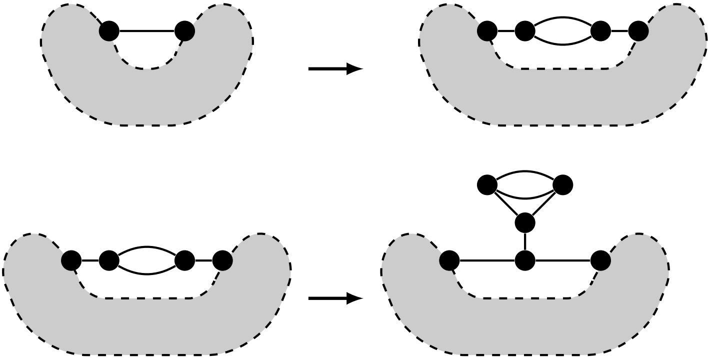


Which are the construction operations?

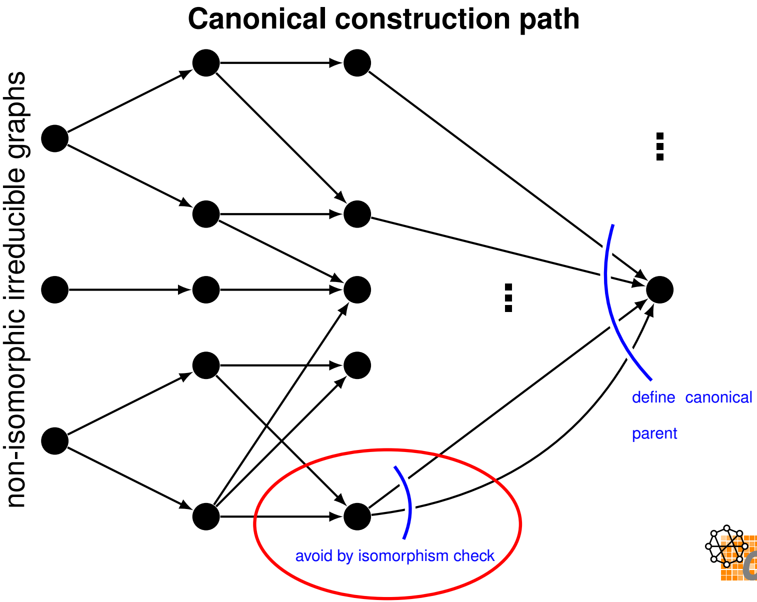




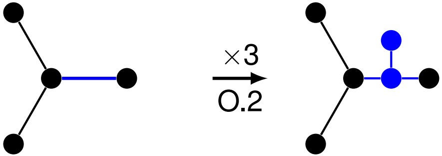
Which are the construction operations?



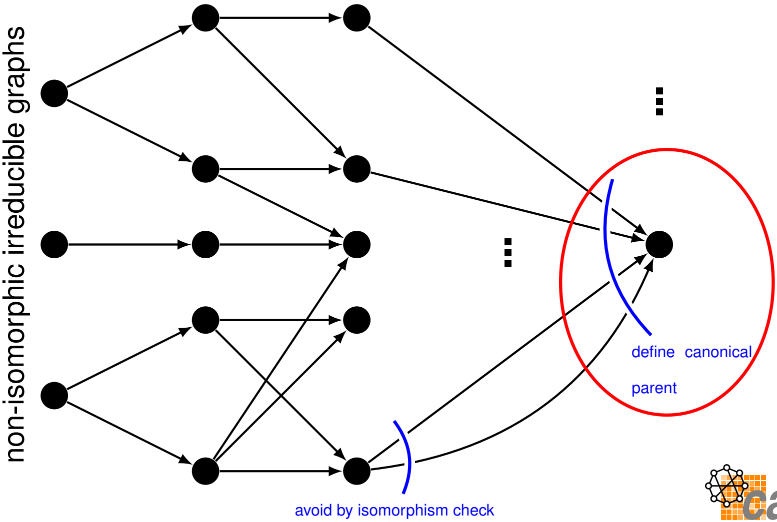
Which technique will be used?



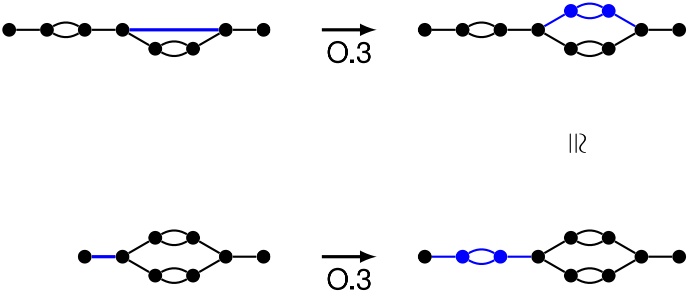
Avoid the same graph from the same parent



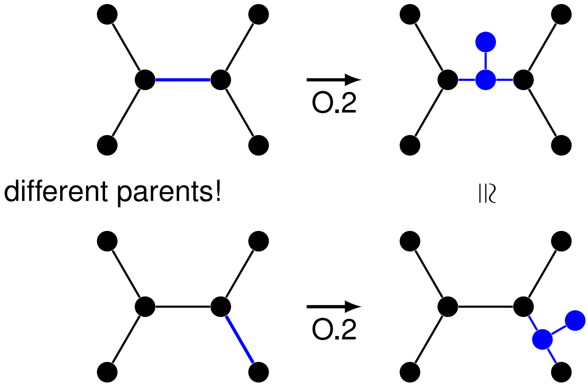
Canonical construction path



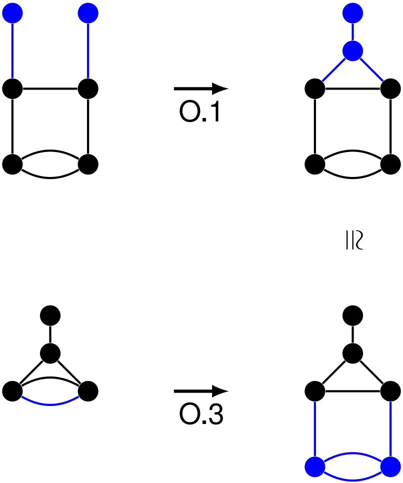
Avoid the same graph from different parents



Avoid the same graph from different parents

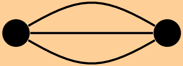


Avoid the same graph from different parents

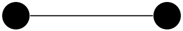
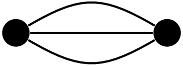
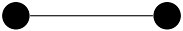
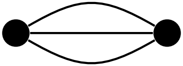


What are the irreducible graphs?

Use inverse operation to reduce each pregraph primitive

$\Rightarrow$ each pregraph primitive can be reduced to a cubic simple graph,  or 

What are the irreducible graphs?

Target class	**Irreducible graphs**
$\mathcal{C}$	$\mathcal{C}$
$\mathcal{G}_{1,3}$	$\mathcal{C}$ 
$\mathcal{M}$	$\mathcal{C}$ 
$\mathcal{P}*$	$\mathcal{C}$  

What are the irreducible graphs?

- degree 1 vertices don't count towards the order of the graph when translating from $\mathcal{G}_{1,3}$ to $\mathcal{S}$ (and similar)
- number of degree 3 vertices never decreases when applying the construction operations

- $\mathcal{L}$,$\mathcal{M}$,$\mathcal{LM}$ with n vertices $\rightarrow \mathcal{C}$ with $\leq n$ vertices.
- $\mathcal{S}$,$\mathcal{LS}$,$\mathcal{SM}$,$\mathcal{LSM}$ with n vertices $\rightarrow \mathcal{C}$ with $\leq n$ vertices, but intermediate $\mathcal{G}_{1,3}$ and $\mathcal{P}*$ with $\leq 2n+2$ vertices

Translation from $\mathcal{G}_{1,3}$ to $\mathcal{L}$, $\mathcal{S}$ and $\mathcal{LS}$

$\mathcal{G}_{1,3}(n)$ to $\mathcal{L}$(n)

translation always possible

$\mathcal{G}_{1,3}(\leq 2n+2)$ to $\mathcal{S}(n)$

$\forall G \in \mathcal{G}_{1,3}(\leq 2n+2) : V_3(G) = n \Rightarrow \exists! G' \in \mathcal{S}(n) : *(G') = G$

$\mathcal{G}_{1,3}(\leq 2n+2)$ to $\mathcal{LS}(n)$

$\forall G \in \mathcal{G}_{1,3}(\leq 2n+2) : V(G) \geq n \wedge V_3(G) \leq n \Rightarrow \exists G' \in \mathcal{LS}(n) :$
$*(G') = G$
$n - V_3(G)$ vertices of degree 1 correspond to vertices with loops, rest corresponds to semi-edges
(homomorphism principle)

Results and timings

	C	L	S	M	LS	LM	SM	LSM
1	0	0	1	0	2	0	1	2
2	0	1	1	1	3	2	3	5
3	0	0	2	0	4	0	4	7
4	1	2	6	2	12	5	12	22
5	0	0	10	0	22	0	22	43
6	2	6	29	6	68	17	68	141
7	0	0	64	0	166	0	166	373
8	5	20	194	20	534	71	534	1270
9	0	0	531	0	1589	0	1589	4053
10	19	91	1733	91	5464	388	5464	14671
11	0	0	5524	0	18579	0	18579	52826
12	85	509	19430	509	68320	2592	68320	203289
13	0	0	69322	0	255424	0	255424	795581
14	509	3608	262044	3608	1000852	21096	1000852	3241367
15	0	0	1016740	0	4018156	0	4018156	13504130
16	4060	31856	4101318	31856	16671976	204638	16671976	57904671
17	0	0	16996157	0	70890940	0	70890940	253856990
18	41301	340416	72556640	340416	309439942	2317172	309439942	1139231977

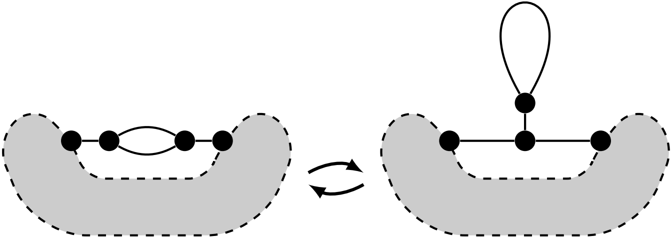
Results and timings

	L	S	M	LS	LM	SM	LSM
1	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s
2	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s
3	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s
4	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s
5	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.004s
6	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.004s	0m0.000s
7	0m0.000s	0m0.004s	0m0.000s	0m0.000s	0m0.000s	0m0.000s	0m0.004s
8	0m0.004s	0m0.004s	0m0.004s	0m0.004s	0m0.000s	0m0.008s	0m0.012s
9	0m0.000s	0m0.012s	0m0.000s	0m0.016s	0m0.000s	0m0.024s	0m0.032s
10	0m0.000s	0m0.044s	0m0.004s	0m0.064s	0m0.004s	0m0.124s	0m0.116s
11	0m0.000s	0m0.152s	0m0.000s	0m0.220s	0m0.000s	0m0.340s	0m0.436s
12	0m0.008s	0m0.572s	0m0.012s	0m0.864s	0m0.028s	0m1.328s	0m1.764s
13	0m0.000s	0m2.244s	0m0.000s	0m3.480s	0m0.000s	0m5.348s	0m7.492s
14	0m0.052s	0m8.861s	0m0.064s	0m14.717s	0m0.244s	0m22.409s	0m32.174s
15	0m0.000s	0m36.390s	0m0.000s	1m5.672s	0m0.000s	1m36.474s	2m25.377s
16	0m0.452s	2m35.818s	0m0.620s	4m51.342s	0m2.704s	7m5.659s	11m14.182s
17	0m0.000s	11m23.471s	0m0.000s	22m47.041s	0m0.000s	32m3.192s	53m26.868s
18	0m5.360s	51m22.321s	0m7.460s	111m38.047s	0m33.438s	148m21.844s	262m1.763s

2.40 GHz Intel Xeon

caagt

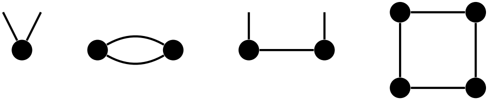
Connection loops and multi-edges



(Past) future work

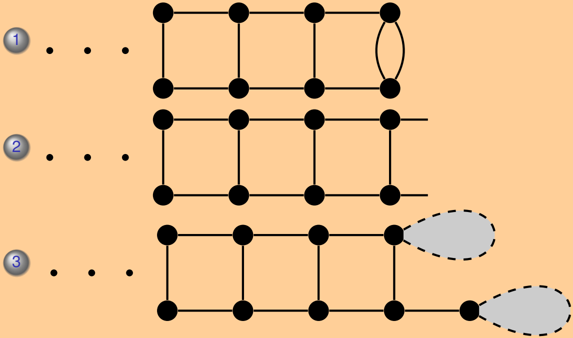
- bipartite graphs
- 3-edge-colorable graphs
- 3-edge-colored graphs
- graphs with a 2-factor where each component is a quotient of a 4-cycle
- graphs with a 2-factor where each component is a 4-cycle
- Specialized generators for the last types

Quotients of a 4-cycle



Filtering these graphs

Does vertex lie in a ladder? (possibly of length 1)



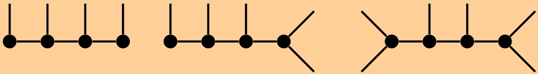
Either side of type 1 or 2 ⇒ allows good 2-factor
Both sides of type 3 ⇒ allows good 2-factor if length is odd

Filtering these graphs

Also possible are a prism or Möbius ladder and some degenerate cases (cf. 4-cycle with a diagonal)

If vertex is not in a ladder: Is this vertex contained in a digon?

Remainder should be forest of paths with semi-edges such that only endpoints may have degree 2 (otherwise all vertices are trivalent)



If both endpoints have degree 2 then number of vertices should be even

2-factor

	quotient of a C_4			a C_4		
	S	**M**	**SM**	**S**	**M**	**SM**
1	1/1		1/1			
2	1/1	1/1	2/2			
3	1/2		2/4			
4	4/6	2/2	9/12	3/6	2/2	5/12
5	3/10		7/22			
6	10/29	3/6	29/68			
7	9/64		27/166			
8	35/194	9/20	107/534	13/194	5/20	20/534
9	34/531		118/1589			
10	100/1733	14/91	398/5464			
11	128/5524		550/18579			
12	382/19430	47/509	1759/68320	75/19430	18/509	128/68320
13	540/69322		2732/255424			
14	1462/262044	106/3608	8250/1000852			
15	2378/1016740		14263/4018156			
16	6313/4101318	353/31856	42087/16671976	641/4101318	81/31856	1168/16671976
17	11334/16996157		79081/70890940			
18	27745/72556640	1067/340416	220865/309439942			

2-factor

	quotient of a C_4			a C_4		
	S	**M**	**SM**	**S**	**M**	**SM**
1	100		100			
2	100	100	100			
3	50		50			
4	66.6667	100	75	50	100	41.6667
5	30		31.8182			
6	34.4828	50	42.6471			
7	14.0625		16.2651			
8	18.0412	45	20.0375	6.7010	25	3.7453
9	6.4030		7.4261			
10	5.7703	15.3846	7.2840			
11	2.3172		2.9603			
12	1.9660	9.2338	2.5746	0.3860	3.5363	0.1874
13	0.7790		1.0696			
14	0.5579	2.9379	0.8243			
15	0.2339		0.3550			
16	0.1539	1.1081	0.2524	0.0156	0.2543	0.0070
17	0.0667		0.1116			
18	0.0382	0.3134	0.0714			

Thank you for your attention