Spherical Tilings by Congruent Quadrangles

Yohji Akama[1] Nico Van Cleemput[2]

[1]Mathematical Institute
Graduate School of Science
Tohoku University

[2]Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics and Computer Science
Ghent University

Spherical tilings

- edges are parts of great circles
- edge-to-edge tiling
- vertex degree ≥ 3

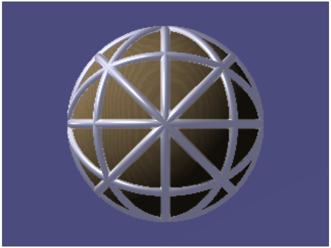
Spherical tilings by congruent polygons

all faces the same size

$$\Downarrow$$

only triangles, quadrangles or pentagons

Classification of spherical tilings by congruent triangles completed by Davies[1] and Ueno-Agaoka[2].



[1] H.L. Davies, "Packings of spherical triangles and tetrahedra", in: *Proc. Colloquium on Convexity (Copenhagen, 1965)*, Kobenhavns Univ. Mat. Inst., 1967, pp. 42–51.

[2] Y. Ueno and Y. Agaoka, "Classification of tilings of the 2-dimensional sphere by congruent triangles", in: *Hiroshima Math. J.* 32.3 (2002), pp. 463–540.

The next step:

classification of spherical tilings by congruent quadrangles

Types of quadrangles

aaaa

abab

aabb

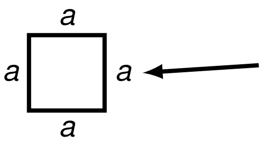
aaab

aabc

abac

abcd

Types of quadrangles



aaaa

abab

aabb

aaab

aabc

abac

abcd

Types of quadrangles

aaaa

abab

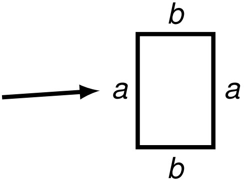
aabb

aaab

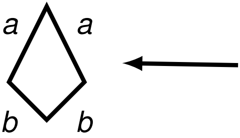
aabc

abac

abcd



Types of quadrangles



aaaa

abab

aabb

aaab

aabc

abac

abcd

Types of quadrangles

aaaa

abab

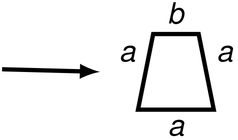
aabb

aaab

aabc

abac

abcd



Types of quadrangles

aaaa

abab

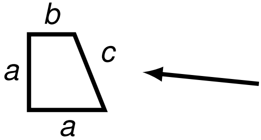
aabb

aaab

aabc

abac

abcd



Types of quadrangles

aaaa

abab

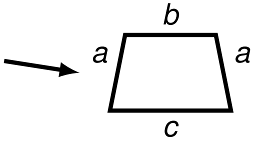
aabb

aaab

aabc

abac

abcd



Types of quadrangles

aaaa

abab

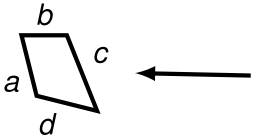
aabb

aaab

aabc

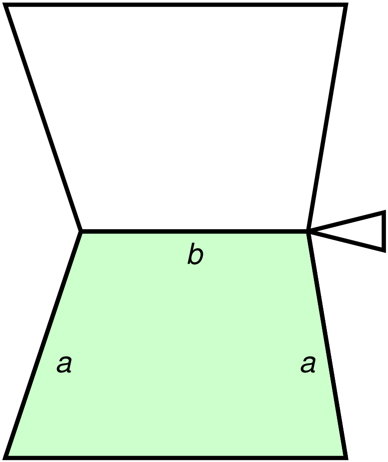
abac

abcd

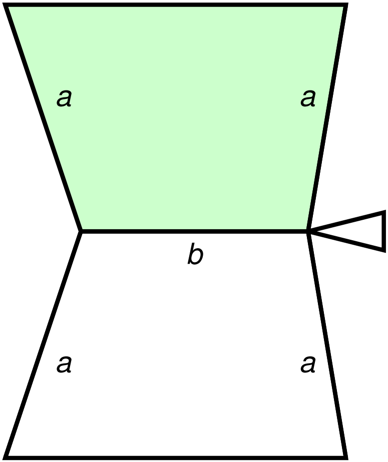


In every quadrangulation of the sphere,
there exists a vertex of degree 3.

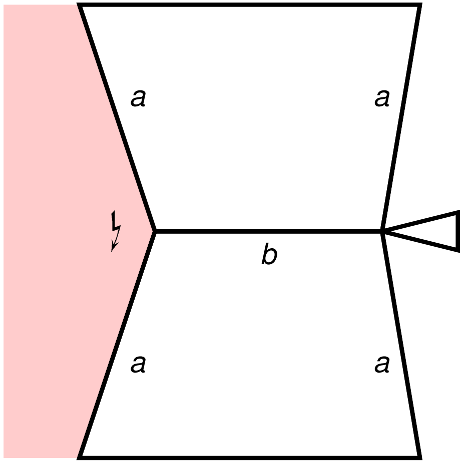
abab, abac



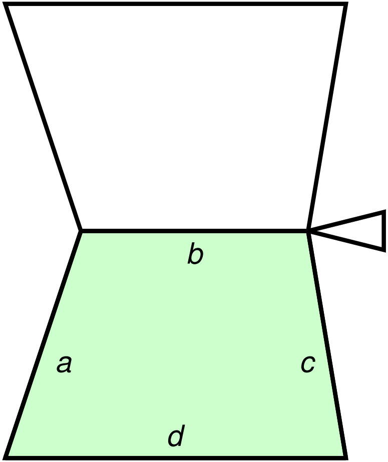
abab, *abac*

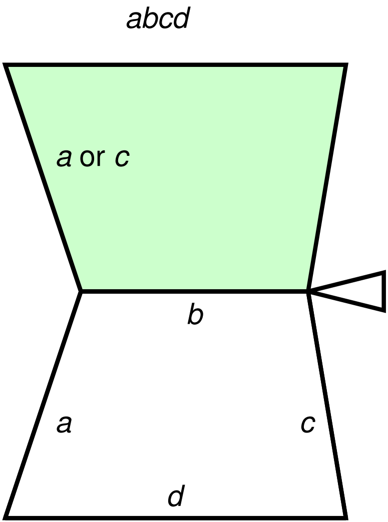


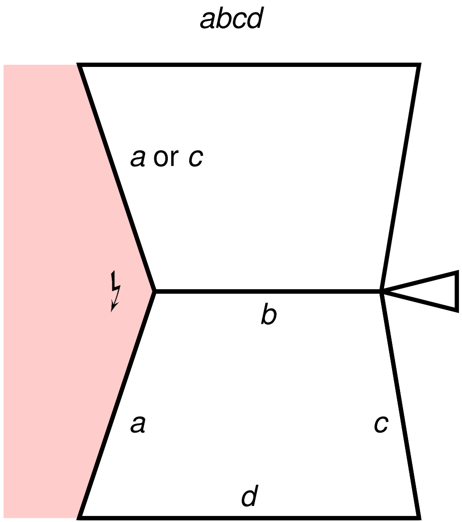
abab, abac



abcd




abcd
a or c
b
a
c
d


abcd
a or c
ϟ
b
a
c
d

Types of quadrangles

aaaa

~~*abab*~~

aaab

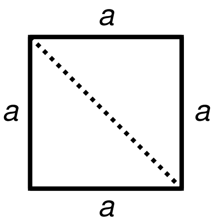
aabb

aabc

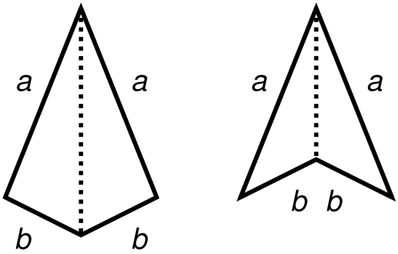
~~*abac*~~

~~*abcd*~~

aaaa



aabb



Classification of spherical tilings by congruent squares, kites and daggers completed by Akama-Sakano[3].

[3] Y. Akama and Y. Sakano, "Classification of spherical tilings by congruent rhombi (kites, darts)", In preparation.

Types of quadrangles

aaaa

~~*abab*~~

aaab

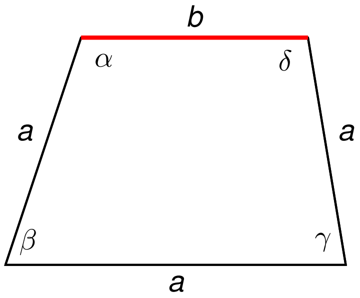
aabb

aabc

~~*abac*~~

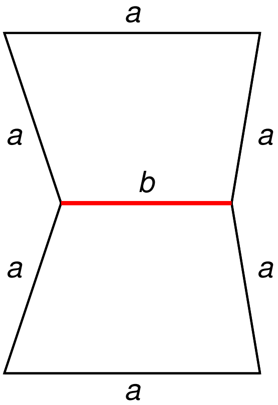
~~*abcd*~~

Type 2 quadrangles



Even number of tiles

Assignment of side lengths corresponds to perfect matching in dual



Concave type 2 quadrangles

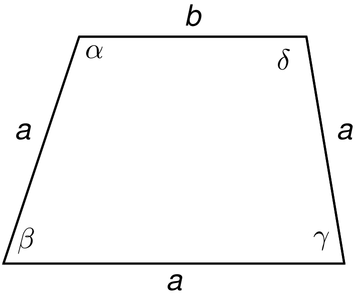
- Ambiguity of inner angles[4]
- Edge which is not a geodesic[4]

[4] Yohji Akama and K. Nakamura, “Spherical tilings by congruent quadrangles over pseudo-double wheels (II) the ambiguity of the inner angles”, Preprint, 2012.

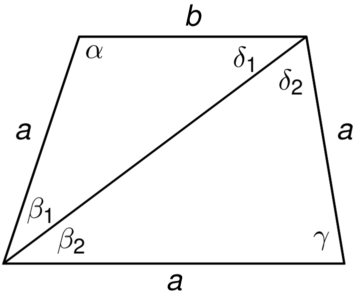
Convex type 2 quadrangles

$$0 < \alpha, \beta, \gamma, \delta < \pi$$

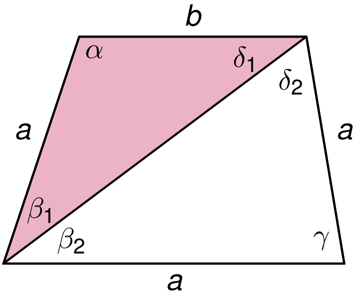
Some restrictions on the angles



Some restrictions on the angles

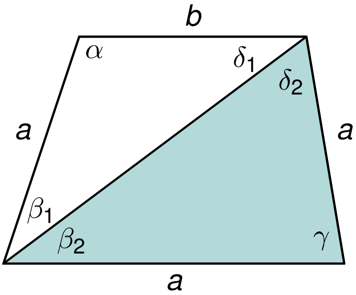


Some restrictions on the angles



$$\alpha + \delta_1 < \pi + \beta_1$$

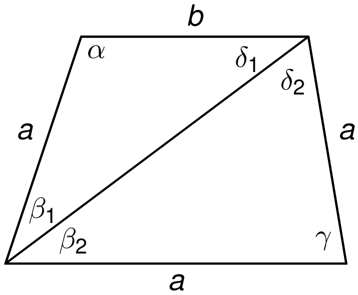
Some restrictions on the angles



$$\alpha + \delta_1 < \pi + \beta_1$$

$$\delta_2 = \beta_2$$

Some restrictions on the angles



$$\alpha + \delta_1 < \pi + \beta_1$$

$$\delta_2 = \beta_2$$

$$\alpha + \delta < \pi + \beta$$

Some restrictions on the angles

$$\alpha + \delta < \pi + \beta$$
$$\alpha + \delta < \pi + \gamma$$

$$\alpha = \delta \Leftrightarrow \beta = \gamma$$

$$(1-\cos\beta)\cos^2\alpha-(1-\cos\beta)(1-\cos\gamma)\cos\alpha\cos\delta+(1-\cos\gamma)\cos^2\delta$$
$$+\cos\beta\cos\gamma+\sin\alpha\sin\beta\sin\gamma\sin\delta=1$$

Area of a tile

$$S = \alpha + \beta + \gamma + \delta - 2\pi$$

$$S = \frac{4\pi}{F}$$

Area of a tile

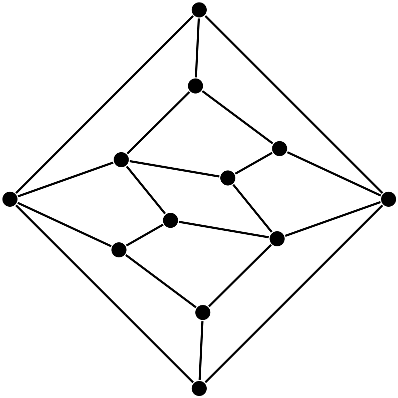
$$S = \alpha + \beta + \gamma + \delta - 2\pi$$

$$S = \frac{4\pi}{F}$$

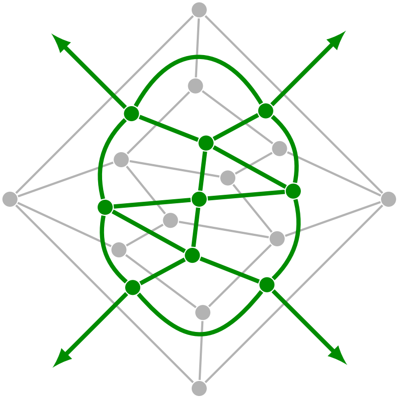
$$\alpha + \beta + \gamma + \delta - 2\pi = \frac{4\pi}{F}$$

Generation of spherical tilings by congruent convex quadrangles of type 2

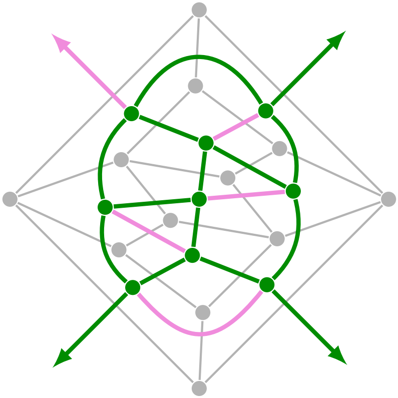
Generate quadrangulations of the sphere



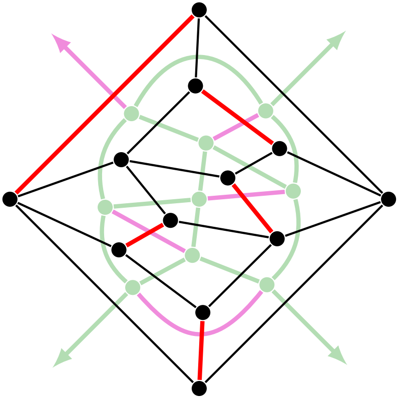
Generate perfect matchings for the dual of the quadrangulation



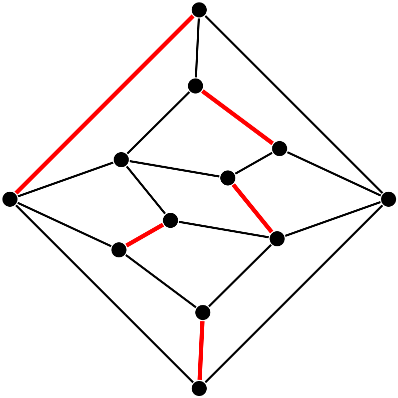
Generate perfect matchings for the dual of the quadrangulation



Generate perfect matchings for the dual of the quadrangulation



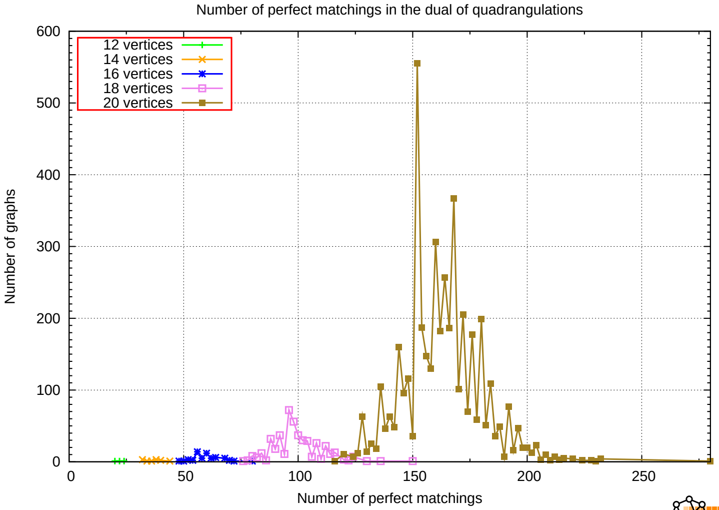
Generate perfect matchings for the dual of the quadrangulation



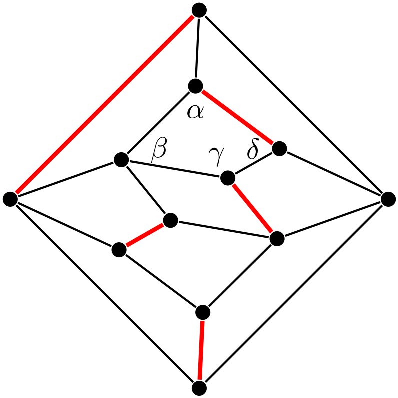
Filter out quadrangulations for which the dual has no perfect matching?

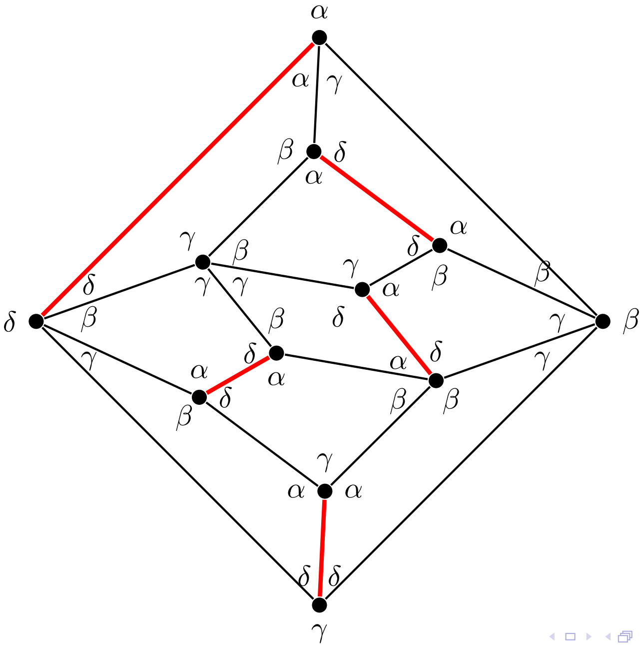
Every edge in the dual of a quadrangulation belongs to a perfect matching of the dual.[5]

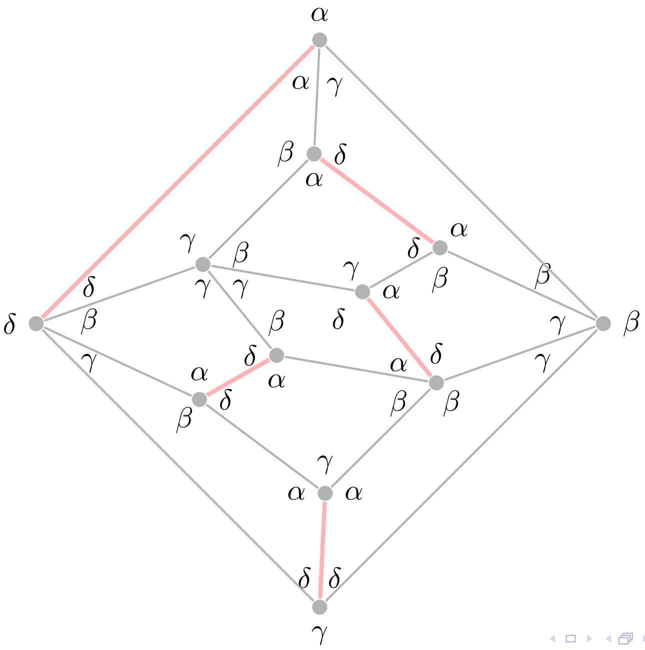
[5] C. D. Carbonera and Jason F. Shepherd, *On the existence of a perfect matching for 4-regular graphs derived from quadrilateral meshes.* Tech. rep., UUSCI-2006-021, SCI Institute Technical Report, 2006.

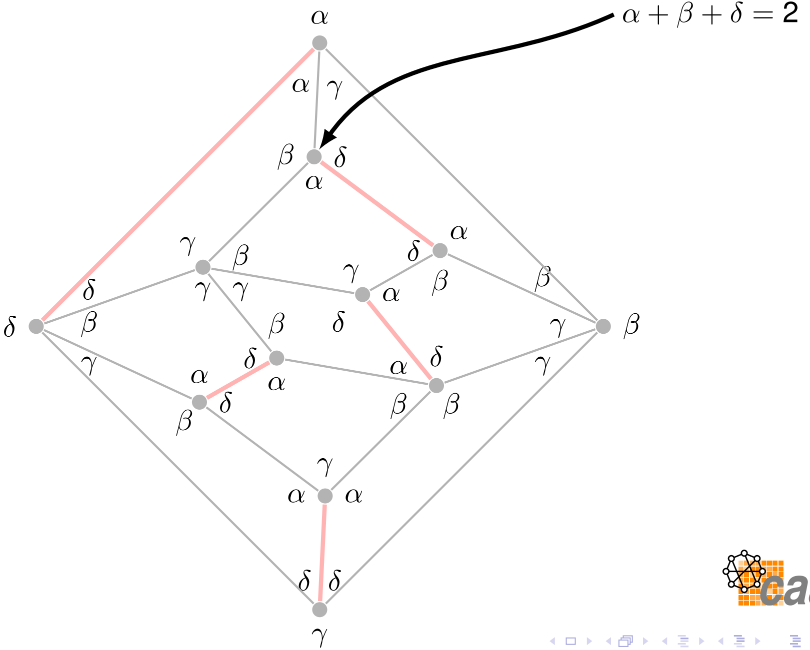

Number of perfect matchings in the dual of quadrangulations
Number of graphs
Number of perfect matchings
12 vertices
14 vertices
16 vertices
18 vertices
20 vertices
0
100
200
300
400
500
600
0
50
100
150
200
250

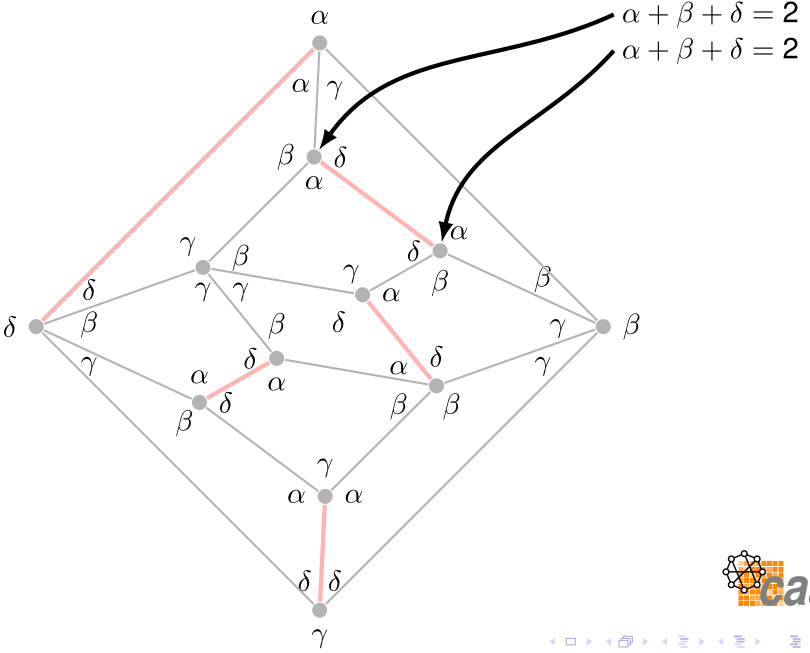
Generate angle assignments: 2^{F-1} possibilities

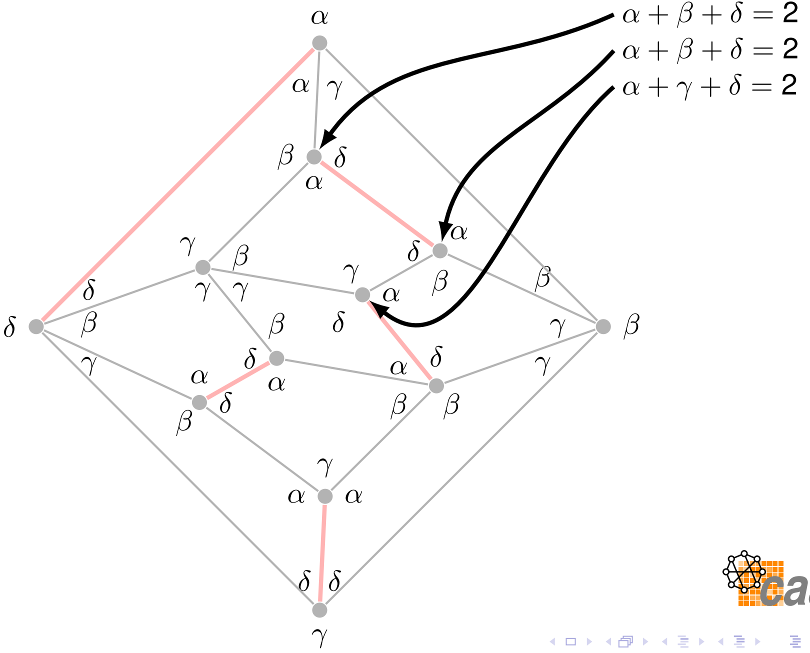


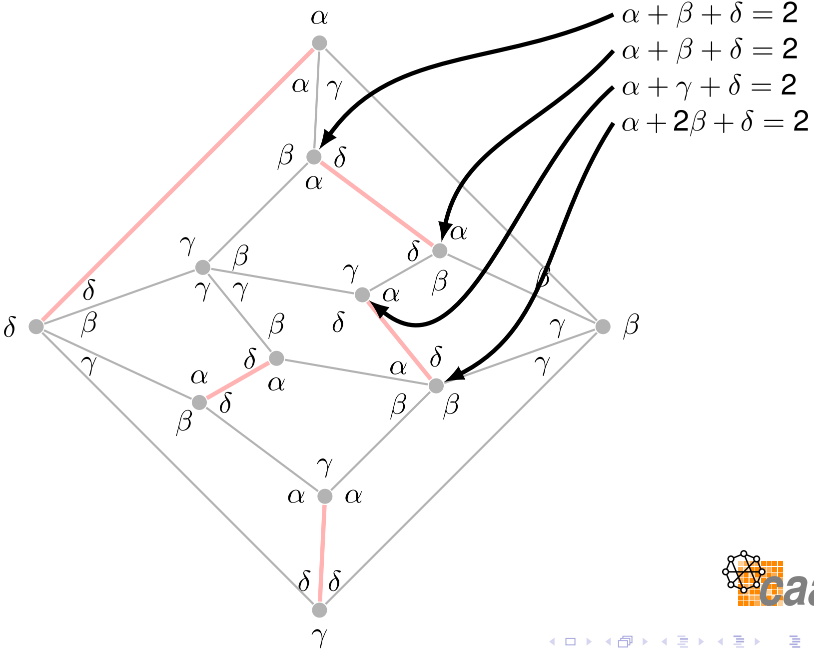


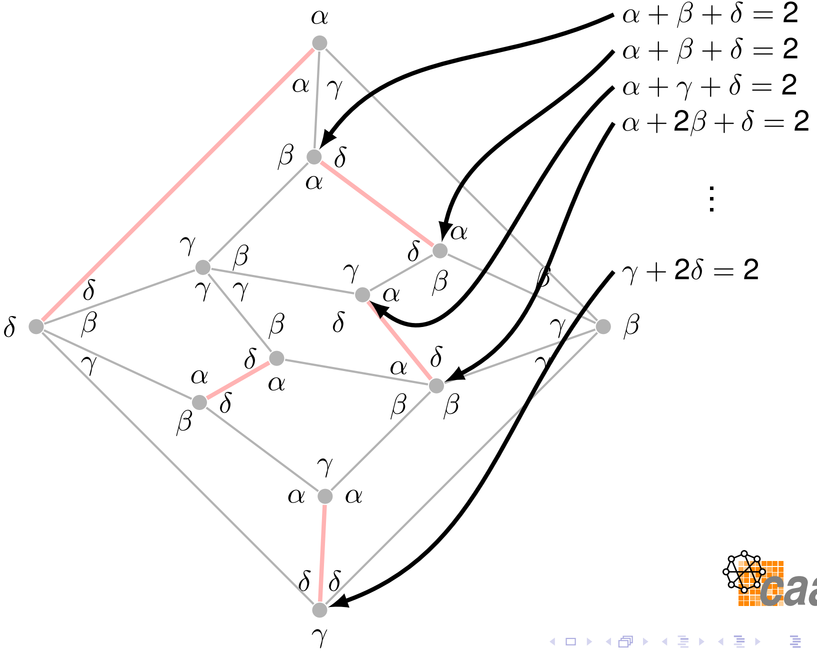

α
α γ
β δ
α
γ β δ α
δ γ γ γ α β β
δ β β δ γ β
γ δ α δ γ
α α
δ β β β
γ
α α
δ δ
γ


$\alpha + \beta + \delta = 2$


$\alpha + \beta + \delta = 2$
$\alpha + \beta + \delta = 2$


$\alpha + \beta + \delta = 2$
$\alpha + \beta + \delta = 2$
$\alpha + \gamma + \delta = 2$


$\alpha + \beta + \delta = 2$
$\alpha + \beta + \delta = 2$
$\alpha + \gamma + \delta = 2$
$\alpha + 2\beta + \delta = 2$



α + β + δ = 2
α + β + δ = 2
α + γ + δ = 2
α + 2β + δ = 2
⋮
γ + 2δ = 2

System of at most V equations:

$$\alpha + \beta + \delta = 2$$
$$\alpha + \gamma + \delta = 2$$
$$\alpha + 2\beta + \delta = 2$$
$$\vdots$$
$$\gamma + 2\delta = 2$$

System of at most V equations:

$$\alpha + \beta + \delta = 2$$
$$\alpha + \gamma + \delta = 2$$
$$\alpha + 2\beta + \delta = 2$$
$$\vdots$$
$$\gamma + 2\delta = 2$$

$$\beta = 0$$

$$0 < \alpha, \beta, \gamma, \delta < 1$$
$$\alpha - \beta + \delta < 1$$
$$\alpha - \gamma + \delta < 1$$
$$\alpha + \beta + \gamma + \delta = 2 + \frac{4}{F}$$

System of vertex equations

$$\alpha = \delta \Leftrightarrow \beta = \gamma$$

$$(1-\cos\beta)\cos^2\alpha-(1-\cos\beta)(1-\cos\gamma)\cos\alpha\cos\delta+(1-\cos\gamma)\cos^2\delta$$
$$+\cos\beta\cos\gamma+\sin\alpha\sin\beta\sin\gamma\sin\delta = 1$$

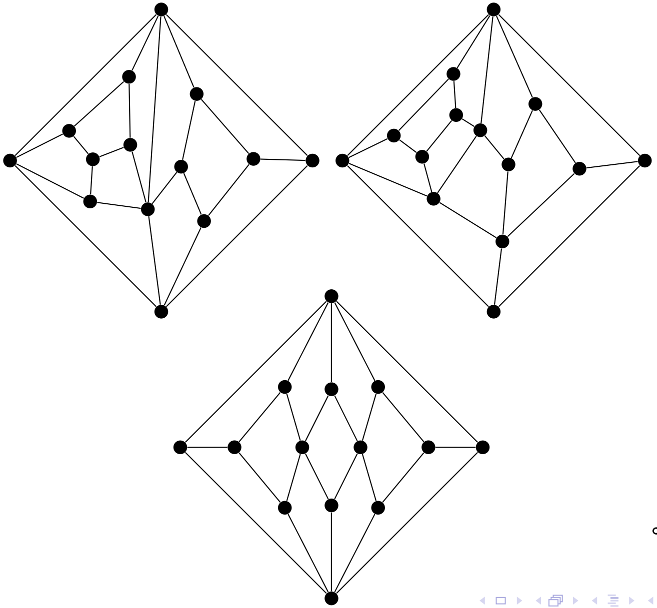
Linear system of equations and inequalities solved with `lp_solve`.

- Freely available (LGPL)
- Easy to integrate in C program

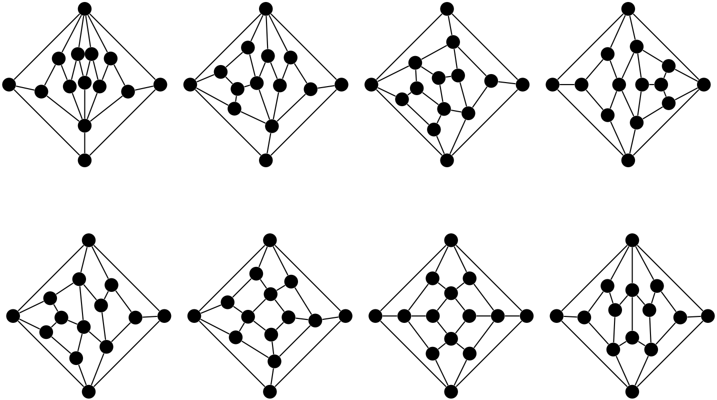
V	Q	M	Assignments	1 coeff diff	lpsolve	solvable
8	1	8	256	0	0	256
10	1	14	1 792	984	520	288
12	3	66	33 792	31 174	2 564	54
14	11	404	827 392	763 363	63 893	136
16	58	3 482	28 524 544	28 088 453	435 972	119
18	451	44698	1 464 664 064	1 458 836 339	5 827 466	259

V	Q	Used	Unused
8	1	1	0
10	1	1	0
12	3	3	0
14	11	8	3
16	58	3	55
18	451	18	433

14 vertices - unused quadrangulations



14 vertices - used quadrangulations



And now...

- Can we exclude some quadrangulation from the start?
- Can we exclude some more systems without using `lp_solve`?
- Is `lp_solve` the best choice?
- Can we include the other restrictions?

$$\alpha = \delta \Leftrightarrow \beta = \gamma$$

$$(1 - \cos\beta)\cos^2\alpha - (1 - \cos\beta)(1 - \cos\gamma)\cos\alpha\cos\delta + (1 - \cos\gamma)\cos^2\delta + \cos\beta\cos\gamma + \sin\alpha\sin\beta\sin\gamma\sin\delta = 1$$

- Do we have all restrictions?