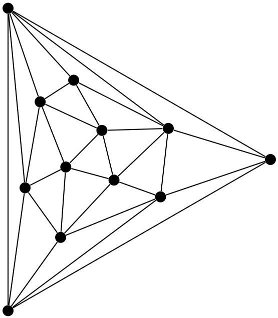
On the strongest form of a theorem of Whitney for hamiltonian cycles in plane triangulations

Gunnar Brinkmann Jasper Souffriau
Nico Van Cleemput

Ghent University

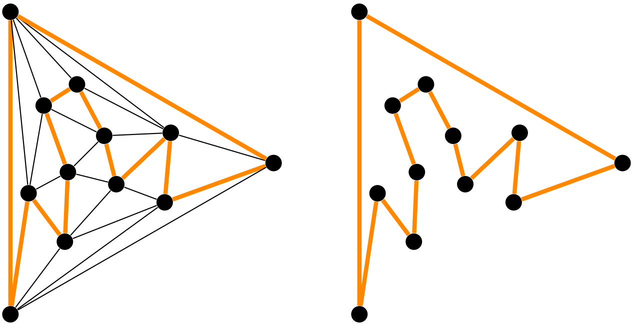
Triangulation

A triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

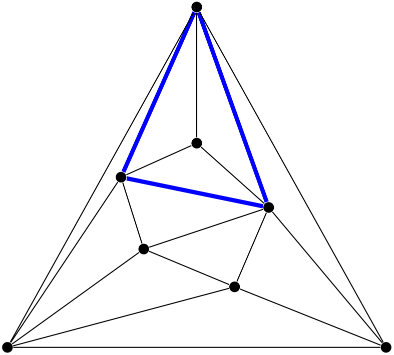
A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.



A graph is hamiltonian if it contains a hamiltonian cycle.

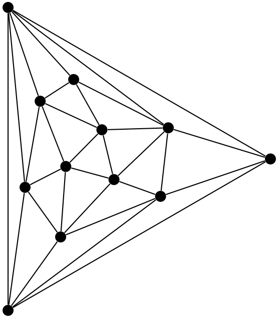
Separating triangles

A separating triangle S in a triangulation T is a subgraph of T such that S is isomorphic to C_3 and $T - S$ has two components.



4-connected triangulations

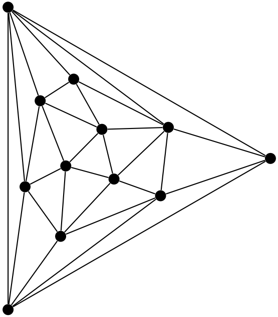
A triangulation is 4-connected if and only if it contains no separating triangles.



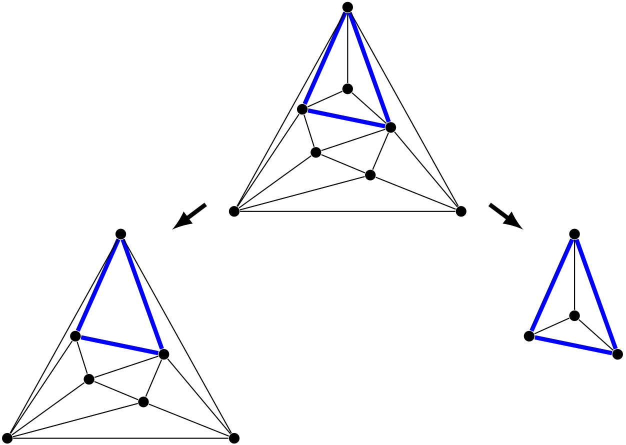
Whitney

Theorem (Whitney, 1931)

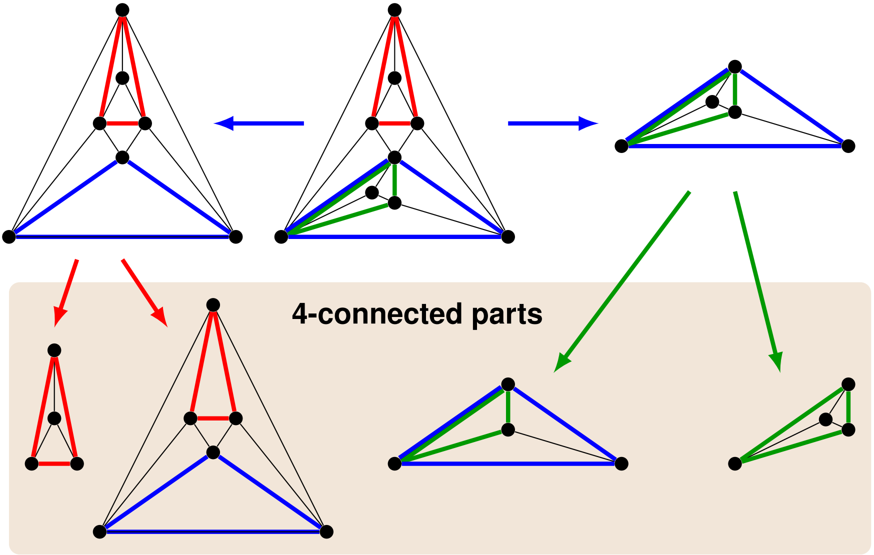
Each triangulation without separating triangles is hamiltonian.



Splitting triangulations

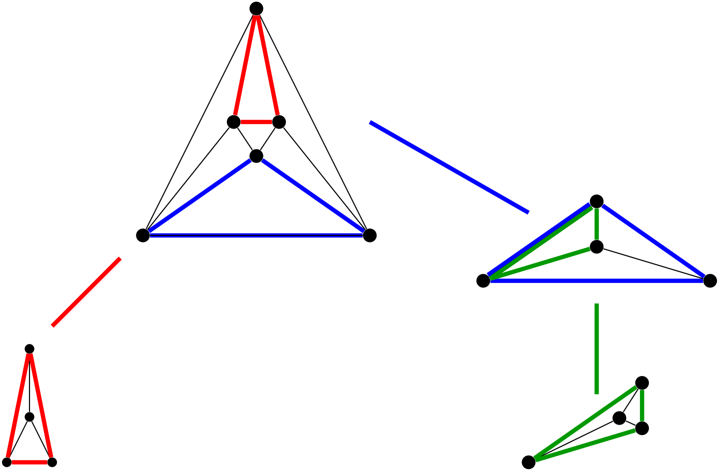


Recursively splitting triangulations



Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles



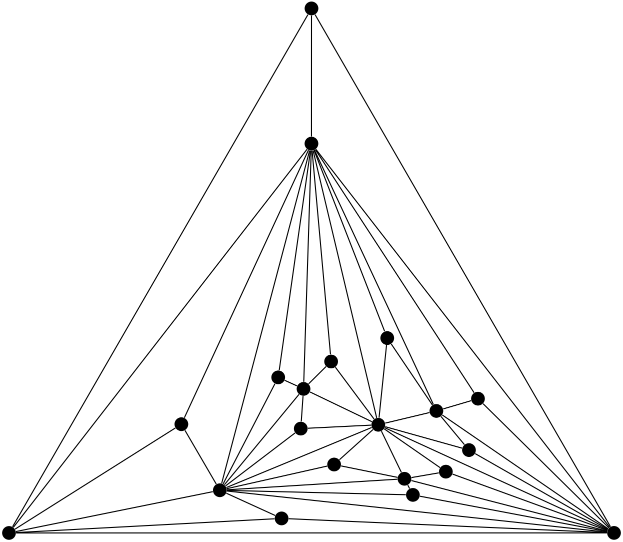
Jackson and Yu

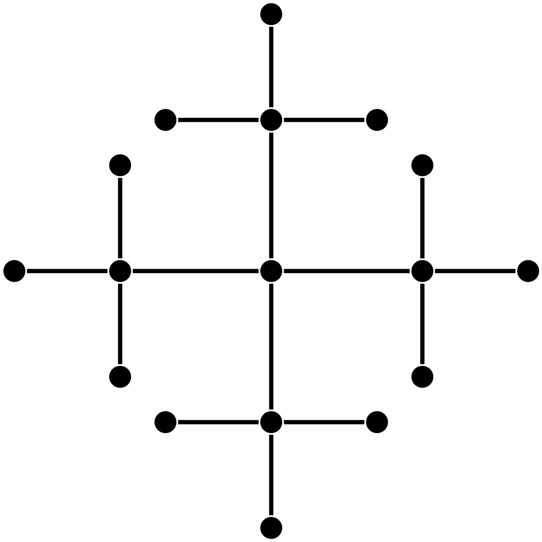
Theorem (Jackson and Yu, 2002)

A triangulation with a decomposition tree with maximum degree 3 is hamiltonian.

Jackson and Yu

There exists a non-hamiltonian triangulation with a decomposition tree with maximum degree 4.



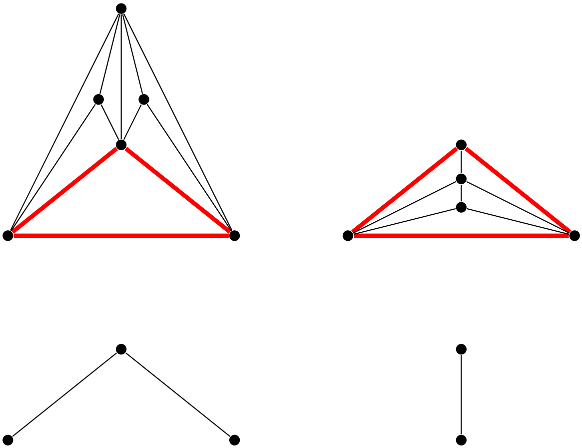


Question

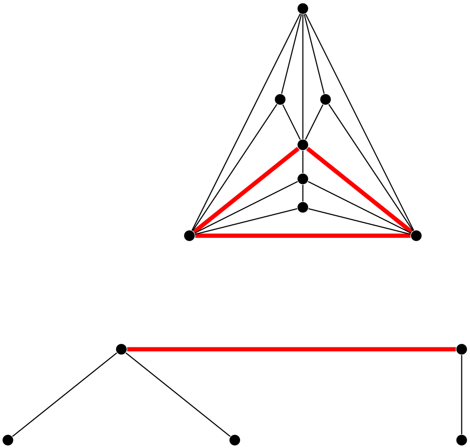
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?

Subdividing a face with a graph



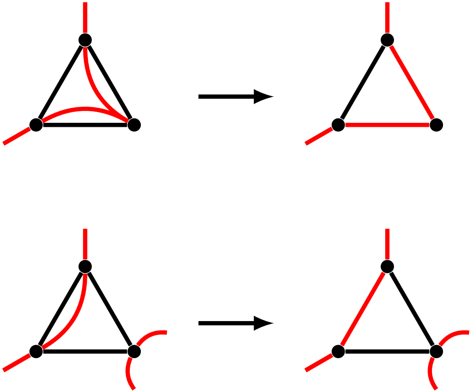
Subdividing a face with a graph



Subdividing a non-hamiltonian triangulation

Lemma

When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.

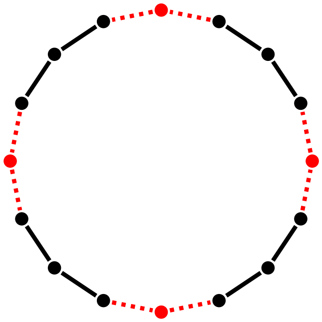


Toughness

A graph is 1-tough if it cannot be split into k components by removing less than k vertices.

Toughness

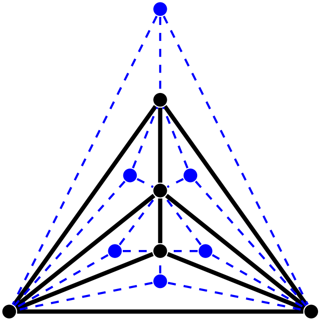
A hamiltonian graph is 1-tough.



Creating a non-hamiltonian plane graph

Lemma

When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.



The subdivided graph is not 1-tough.

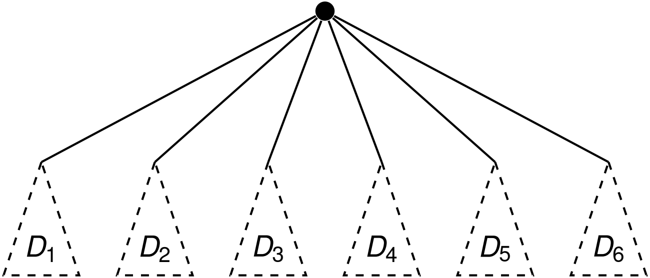
Decomposition trees with $\Delta \geq 6$

Theorem

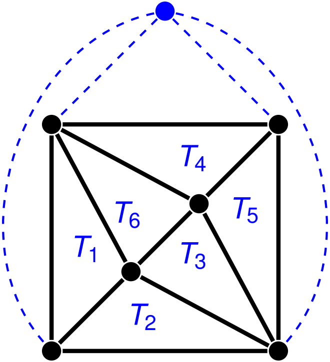
For each tree D with $\Delta(D) \geq 6$, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.

Constructive proof.

Assume $\Delta(D) = 6$.

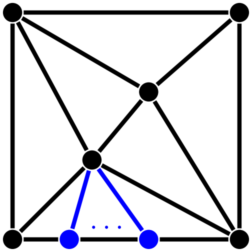


Choose triangulation T_i with decomposition tree D_i $(1 \leq i \leq 6)$



A non-hamiltonian triangulation with D as decomposition tree.

$\Delta(D) > 6$



Remaining cases

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

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Not the decomposition tree of non-hamiltonian triangulation

Remaining cases

Δ : 0 1 2 3 | 4 5 | 6 7 $\cdots$

Not the decomposition tree of non-hamiltonian triangulation (0 1 2 3)

Possibly the decomposition tree of non-hamiltonian triangulation (6 7 …)

Remaining cases

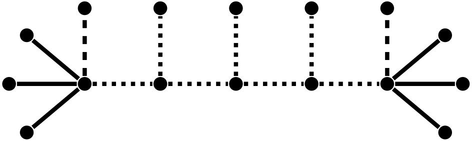
$$\Delta : \; 0 \;\; 1 \;\; 2 \;\; 3 \;\; 4 \;\; 5 \;\; 6 \;\; 7 \;\; \cdots$$

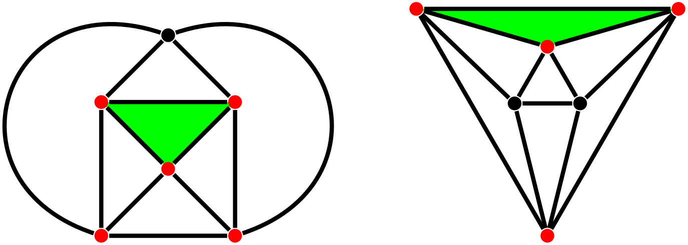
- 0–3: Not the decomposition tree of non-hamiltonian triangulation
- 4–5: ?
- 6, 7, …: Possibly the decomposition tree of non-hamiltonian triangulation

Multiple degrees > 3

Theorem

For each tree D with at least two vertices with degree > 3, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.





red vertices: $5 + (k - 1) + (5 - 3) = 6 + k$
components: $4 + (k - 1) + 4 = 7 + k$

Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.

One vertex of degree 4 or 5

Theorem

Let T be a triangulation with decomposition tree D with only one vertex of degree 4 or 5 and all other vertices of degree at most 3. Then T is 1-tough.

Simplifying things

Theorem

For each $k \geq 4$. Let D be a tree with one vertex of degree k and all other vertices of degree ≤ 3.
There exists a non-hamiltonian triangulation with D as decomposition tree if and only if there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree.

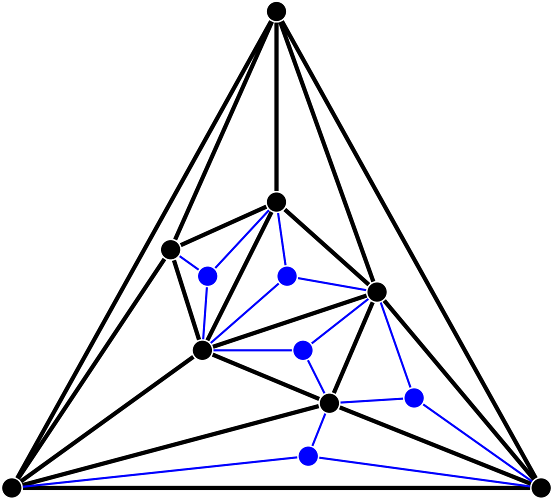
Simplifying things (more)

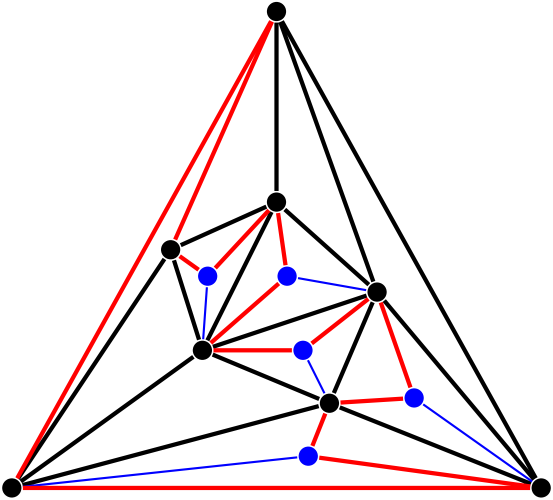
Theorem

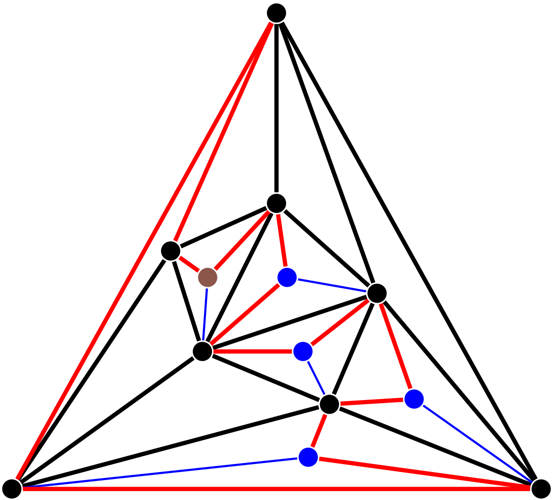
For each $k \geq 4$. If there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree, then there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree such that the leaves correspond to K_4's.

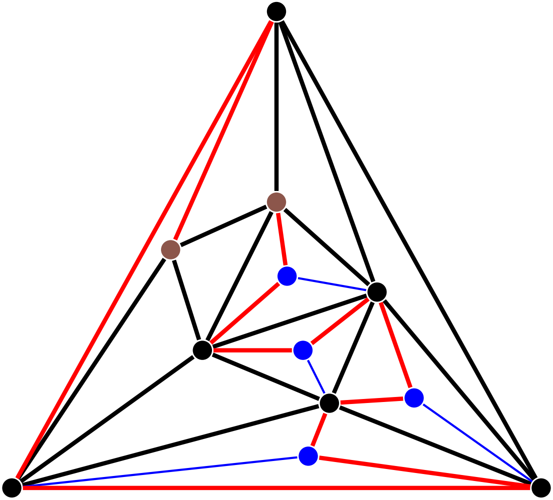
Specialised search

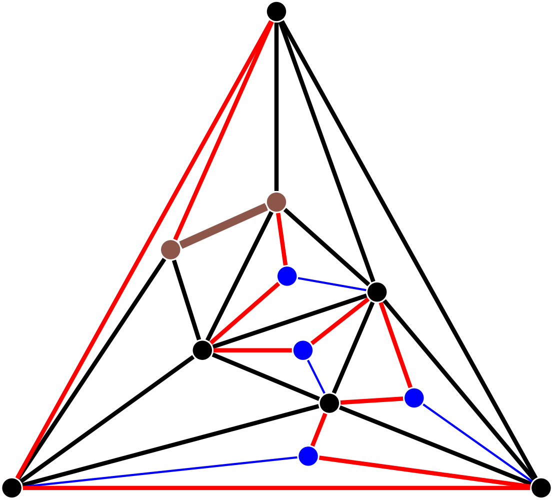
Specialised programs to search for non-hamiltonian triangulations with $K_{1,4}$ or $K_{1,5}$ as decomposition tree.

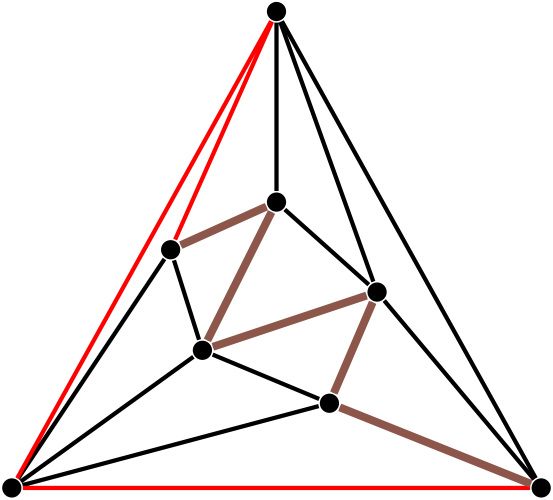


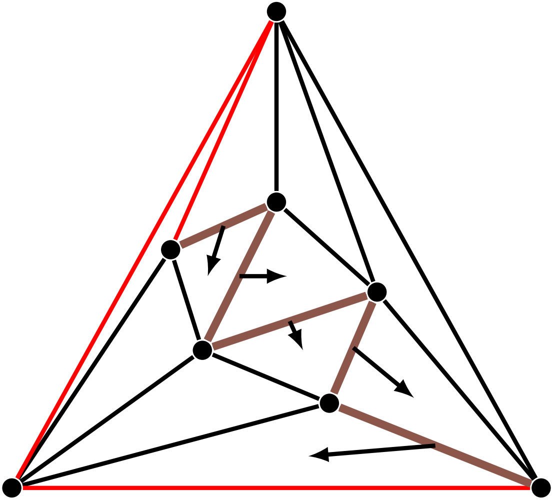












Extended outerplanar discs

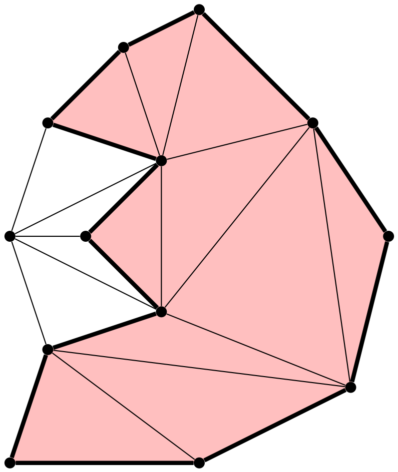
Outerplanar disc of a triangulation T
vertex induced subgraph of T which is an outerplanar triangulation of the disc with at least 3 vertices

Leaf of an outerplanar disc O
vertex which only has two neighbours in O, together with those two neighbours

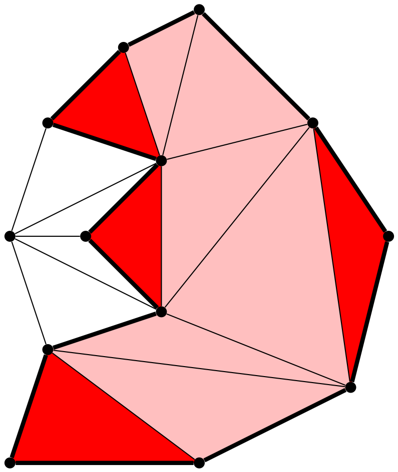
Extended outerplanar disc of a triangulation T
outerplanar disc of T together with a triangle t not belonging to O, but sharing an edge with O

Leaf of an extended outerplanar disc O
Leaf of the outerplanar disc which contains a vertex of degree 2 in O which does not belong to t

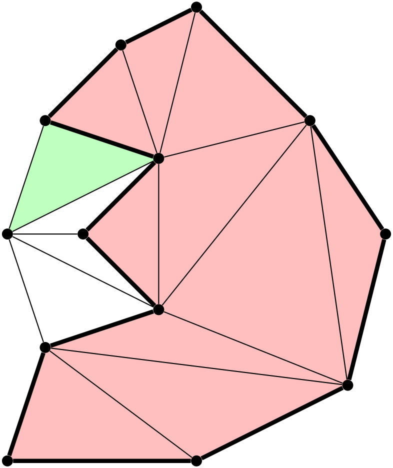
Extended outerplanar discs



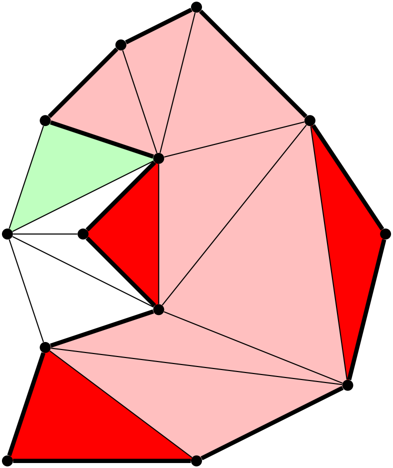
Extended outerplanar discs



Extended outerplanar discs



Extended outerplanar discs



Extended outerplanar discs

Theorem

Let T be a 4-connected triangulation. Let $t_1, t_2, \ldots t_k$ (with $k \geq 4$) be distinct triangles in T such that there is an extended outerplanar disc O with extension t_1 containing $t_2, \ldots, t_{k-2}$ as leaves and with t_{k-1}, t_k not in O.
Then there exists a hamiltonian cycle C of T and edges $e_1 \in E(t_1), \ldots, e_k \in E(t_k)$ that are pairwise distinct and contained in $E(C)$.

Extended outerplanar discs

Corollary

*Let T be a 4-connected plane triangulation, O an extended outerplanar disc and t_1, t_2, t_3, t_4 distinct triangles in G.
If at least two of t_1, t_2, t_3, t_4 are contained in O then there exists a hamiltonian cycle C of T and edges $e_1 \in E(t_1), \ldots, e_4 \in E(t_4)$ that are pairwise distinct and contained in $E(C)$.*

Corollary

*Let T be a 4-connected plane triangulation. Let t_1, t_2, t_3, t_4 be distinct triangles in G such that (at least) two of them share a vertex.
Then there exists a hamiltonian cycle C of T and edges $e_1 \in E(t_1), \ldots, e_4 \in E(t_4)$ that are pairwise distinct and contained in $E(C)$.*

Results

All triangulations on at most 31 vertices with $K_{1,4}$ as decomposition tree are hamiltonian.

All triangulations on at most 27 vertices with $K_{1,5}$ as decomposition tree are hamiltonian.

Results

V	F	4-connected triangulations
6	8	1
7	10	1
8	12	2
9	14	4
10	16	10
11	18	25
12	20	87
13	22	313
14	24	1357
15	26	6244
16	28	30 926
17	30	158 428
18	32	836 749
19	34	4 504 607
20	36	24 649 284
21	38	136 610 879
22	40	765 598 927
23	42	4 332 047 595
24	44	24 724 362 117
25	46	142 205 424 580
26	48	823 687 567 019
27	50	4 801 749 063 379

... and now?

Prove that for each 4-tuple of vertex-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.

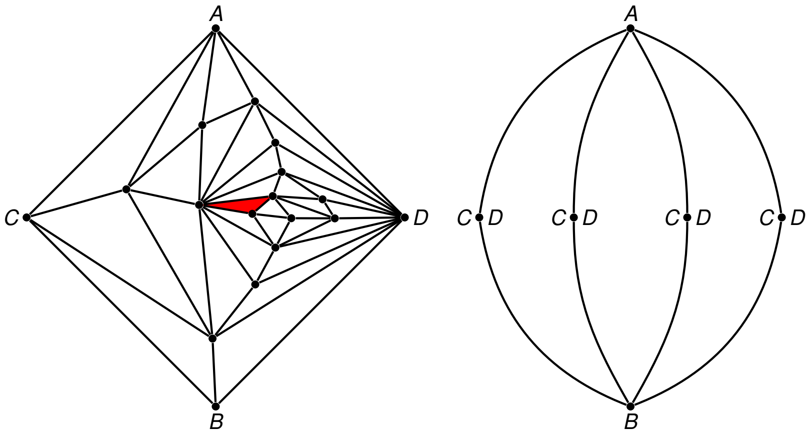
... and now?

Prove that for each 5-tuple of triangles T_1, T_2, T_3, T_4, T_5 in a 4-connected triangulation there exists a hamiltonian cycle C and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

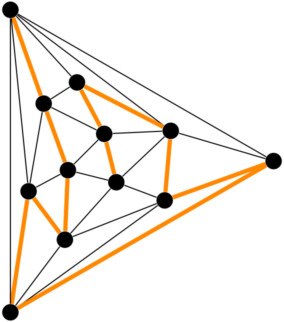
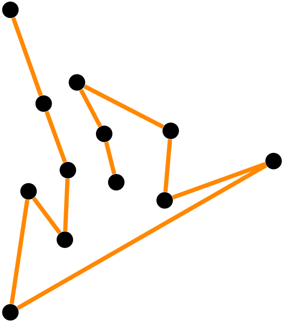
Find a counterexample.

Is there always an eOPD?



Hamiltonian path

A hamiltonian path in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $P_{|V|}$.

Hamiltonian path

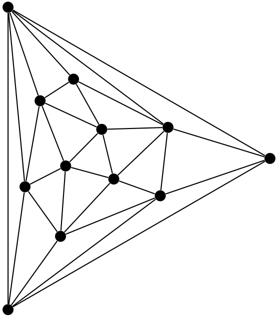
A hamiltonian path connecting x and y is a hamiltonian path P such that x and y have degree 1 in P.

A graph $G(V, E)$ is hamiltonian-connected if for each pair x, y of distinct vertices in V there exists a hamiltonian path connecting x and y.

4-connected triangulations

Theorem (Thomassen, 1983)

Each triangulation without separating triangles is hamiltonian-connected.



3-connected triangulations

Theorem

Let G be a 3-connected triangulation such that there is an edge e which is contained in all separating triangles. Then G is hamiltonian-connected.

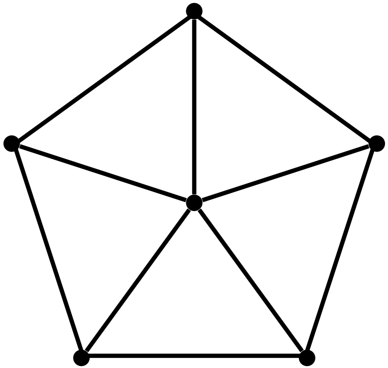
Corollary

Let G be a 3-connected triangulation with exactly one separating triangle. Then G is hamiltonian-connected.

3-connected triangulations

Theorem

For any $s \geq 4$ there exists a 3-connected triangulation with exactly s separating triangles that is not hamiltonian-connected.



Decomposition tree

Theorem

Let D be a tree with maximum degree 1. Then any triangulation which has D as decomposition tree is hamiltonian-connected.

Theorem

Let D be a tree with maximum degree at least 4. Then D is the decomposition tree of a 3-connected triangulation which is not hamiltonian-connected.

Separating triangles

Lemma

On up to 22 vertices all triangulations with at most 2 separating triangles are all hamiltonian-connected.

Lemma

On up to 21 vertices all triangulations with at most 3 separating triangles are all hamiltonian-connected.

Decomposition tree

Lemma

On up to 21 vertices all triangulations that have a decomposition tree with maximum degree 2 are all hamiltonian-connected.

Lemma

On up to 20 vertices all triangulations that have a decomposition tree with maximum degree 3 are all hamiltonian-connected.

Summary for hamiltonian-connectedness

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

Summary for hamiltonian-connectedness

$$\Delta : \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

Always hamiltonian-connected

Summary for hamiltonian-connectedness

Δ : 0 1 2 3 4 5 6 7 $\cdots$

Always hamiltonian-connected

Possibly not hamiltonian-connected

Summary for hamiltonian-connectedness