On the number of hamiltonian cycles in triangulations with few separating triangles

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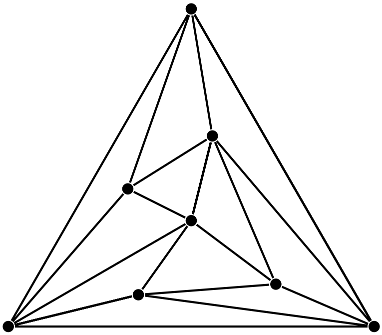
Outline

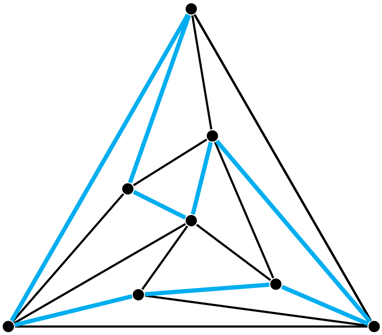
Plane triangulation

A (plane) triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

A hamiltonian cycle C in a graph $G = (V, E)$ is a spanning subgraph of G which is isomorphic to a cycle.

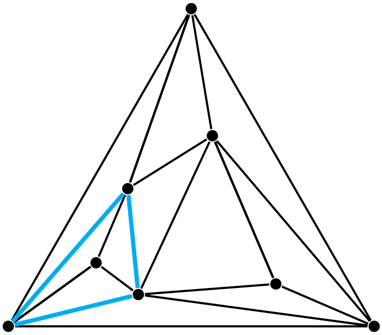


Hamiltonian cycle

$\mathcal{C}$: set of all hamiltonian cycles in graph G

Separating triangle

A separating triangle S in a triangulation G is a subgraph of G which is isomorphic to C_3 such that $G - S$ has two components.



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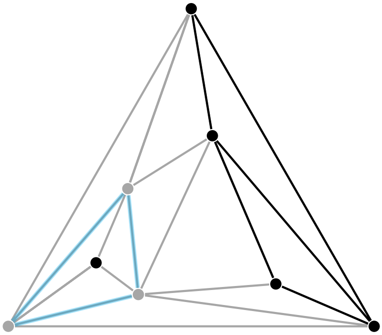
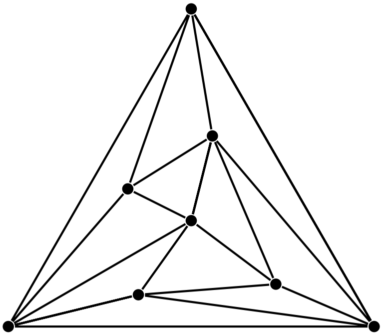


4-connected triangulation

A triangulation on $n > 4$ vertices is 4-connected if and only if it contains no separating triangles.



Lower bound on number of hamiltonian cycles

Theorem (Whitney, 1931)

Every 4-connected triangulation is hamiltonian (i.e., contains at least one hamiltonian cycle).

Theorem (Jackson and Yu, 2002 (reformulated))

Every triangulation with at most 3 separating triangles is hamiltonian (i.e., contains at least one hamiltonian cycle).

Lower bound on number of hamiltonian cycles

Theorem (Kratochvíl and Zeps, 1988)

Every hamiltonian triangulation on at least 5 vertices contains at least four hamiltonian cycles.

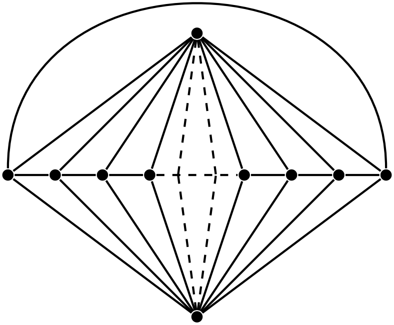
Theorem (Hakimi, Schmeichel and Thomassen, 1979)

Every 4-connected triangulation on n vertices contains at least $\frac{n}{\log_2 n}$ hamiltonian cycles.

Lower bound on number of hamiltonian cycles

Conjecture (Hakimi, Schmeichel and Thomassen, 1979)

Every 4-connected triangulation on n vertices contains at least $2(n-2)(n-4)$ *hamiltonian cycles.*

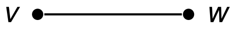


Lower bound on number of hamiltonian cycles

Theorem (Hakimi, Schmeichel and Thomassen, 1979)

Every 4-connected triangulation on n vertices contains at least $\frac{n}{\log_2 n}$ hamiltonian cycles.

Proof by Hakimi, Schmeichel and Thomassen



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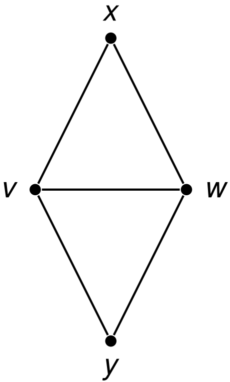

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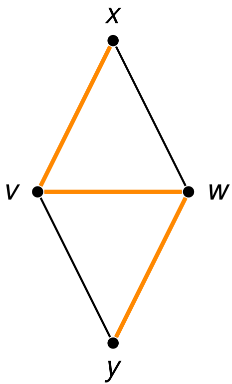


Zigzag

Proof by Hakimi, Schmeichel and Thomassen

For each edge *vw* in *G*: pick hamiltonian cycle containing *xvwy*.

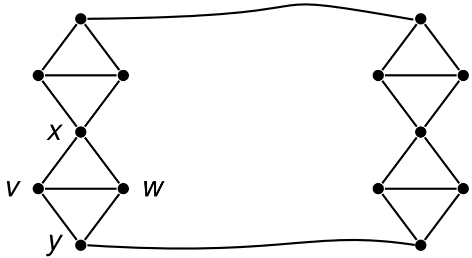
$\Rightarrow \leq 3n - 6$ hamiltonian cycles.

Each hamiltonian cycle occurs at most α times.

$\Rightarrow |\mathcal{C}| \geq \frac{3n-6}{\alpha}$

Proof by Hakimi, Schmeichel and Thomassen

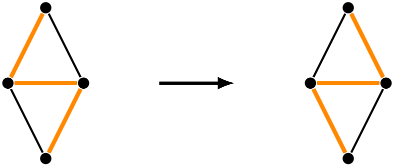
Let C be hamiltonian cycle that occurs α times.



At least $\frac{\alpha}{3}$ zigzags intersect in at most one vertex.

Proof by Hakimi, Schmeichel and Thomassen

New hamiltonian cycle for each independent zigzag switch.



$\Rightarrow |\mathcal{C}| \geq 2^{\frac{\alpha}{3}}$

Proof by Hakimi, Schmeichel and Thomassen

$$\log_2 |\mathcal{C}| \geq \frac{\alpha}{3} \geq \frac{n-2}{|\mathcal{C}|}$$

$$\Downarrow$$

$$|\mathcal{C}| \log_2 |\mathcal{C}| \geq n - 2$$

$$\Downarrow$$

$$|\mathcal{C}| > \frac{n}{\log_2 n}$$

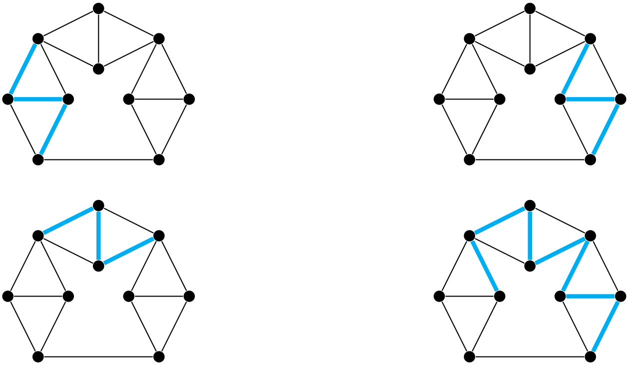
Outline

Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

General technique for finding a lower bound for the size of an arbitrary set $\mathcal{C}' \subseteq \mathcal{C}$ of hamiltonian cycles in a given graph G.

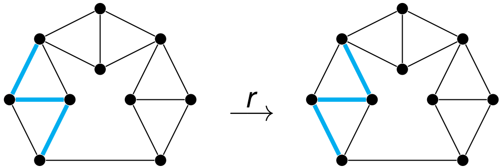
Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

- $\mathcal{S} \subseteq \{$subgraphs of $G\}$



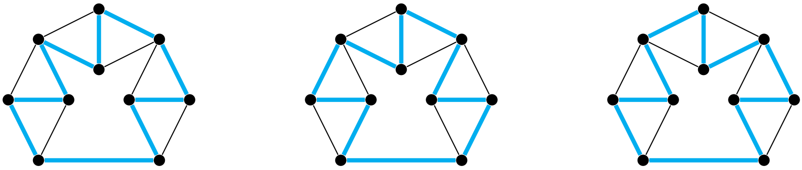
Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

- $\mathcal{S} \subseteq \{\text{subgraphs of } G\}$
- $r : \mathcal{S} \to \{\text{subgraphs of } G\}$



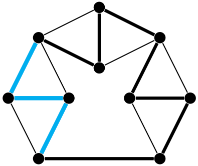
Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

- $\mathcal{S} \subseteq \{\text{subgraphs of } G\}$
- $r : \mathcal{S} \rightarrow \{\text{subgraphs of } G\}$
- $\mathcal{C}' \subseteq \mathcal{C}$



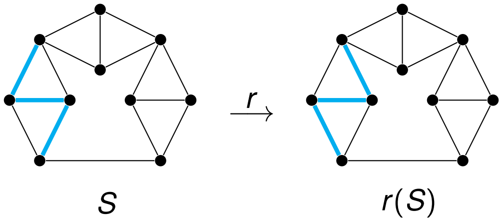
Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

1 Each $S \in \mathcal{S}$ must be contained in at least one $C \in \mathcal{C}'$.



Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

1. Each $S \in \mathcal{S}$ must be contained in at least one $C \in \mathcal{C}'$.
2. For each $S \in \mathcal{S}$ we have that $S \not\subseteq r(S)$.



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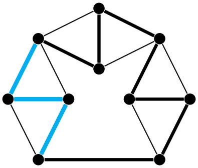
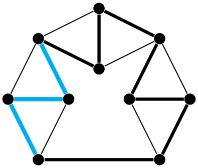
3. For each $S \in \mathcal{S}$ and $C \in \mathcal{C}'$ with $S \subseteq C$ we have that $(C \backslash S) \cup r(S) \in \mathcal{C}'$.



C



$(C \backslash S) \cup r(S)$

Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

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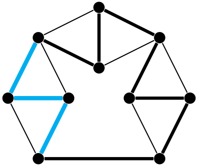
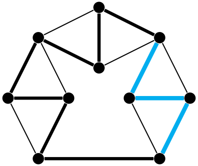
4. For two different $S_1, S_2 \in \mathcal{S}$ and for any $C \in \mathcal{C}'$ containing both subgraphs we have that $(C \backslash S_1) \cup r(S_1) \neq (C \backslash S_2) \cup r(S_2)$.



$S_1 \subseteq C$



$S_2 \subseteq C$

Counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$ in G

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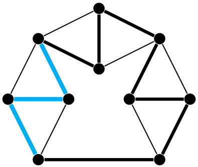
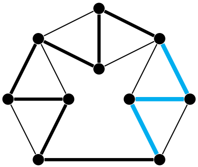


$(C \backslash S_1) \cup r(S_1)$



$(C \backslash S_2) \cup r(S_2)$

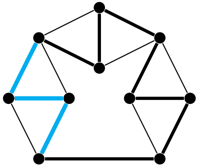
Overlap $o_{\mathcal{S}}(C, X)$ and maximum overlap $O_{\mathcal{S}}(\mathcal{C}', r)$

- $\bar{\mathcal{S}} = \mathcal{S} \cup \{r(S) | S \in \mathcal{S}\}$

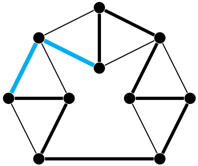
Overlap $o_{\mathcal{S}}(C, X)$ and maximum overlap $O_{\mathcal{S}}(\mathcal{C}', r)$

- $\bar{\mathcal{S}} = \mathcal{S} \cup \{r(S) | S \in \mathcal{S}\}$
- For each $X \in \bar{\mathcal{S}}$ and each $C \in \mathcal{C}'$ with $X \subseteq C$:

$$o_{\mathcal{S}}(C, X) = |\{S \in \mathcal{S} \mid X \cap S \neq \emptyset \text{ and } S \subseteq C\}|$$



$X \subseteq C$



$S \subseteq C$

Overlap $o_{\mathcal{S}}(C, X)$ and maximum overlap $O_{\mathcal{S}}(\mathcal{C}', r)$

- $\bar{\mathcal{S}} = \mathcal{S} \cup \{r(S) | S \in \mathcal{S}\}$
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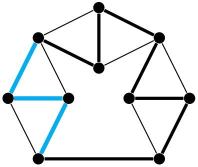

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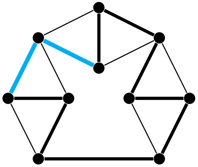


$S \subseteq C$

- $O_{\mathcal{S}}(\mathcal{C}', r) = \max\{o_{\mathcal{S}}(C, X) \mid C \in \mathcal{C}', X \in \bar{\mathcal{S}} : X \subseteq C\}$

Theorem (Brinkmann, Souffriau, NVC, 2014)

Given a graph G, a set $\mathcal{C}' \subseteq \mathcal{C}$, and a nonempty counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$, then

$$|\mathcal{C}'| \geq \frac{|\mathcal{S}|}{O_{\mathcal{S}}(\mathcal{C}', r)}.$$

Zigzags

- Let G be a 4-connected triangulation
- Let $\mathcal{S}$ be the set of all zigzags ($|\mathcal{S}| = 6n - 12$)
- r switches zigzag to mirror image ($\bar{\mathcal{S}} = \mathcal{S}$)
- $\mathcal{C}' = \mathcal{C}$
- $o_{\mathcal{S}}(C, X) \leq 5$, so $O_{\mathcal{S}}(\mathcal{C}, r) \leq 5$

$$\Rightarrow |\mathcal{C}| \geq \frac{6n-12}{O_{\mathcal{S}}(\mathcal{C},r)} \geq \frac{6n-12}{5}$$

Root path and inverse root path

Theorem (Brinkmann, Souffriau, NVC, 2014)

Every 4-connected triangulation on n vertices has $12(n-2)$ *root paths and inverse root paths.*

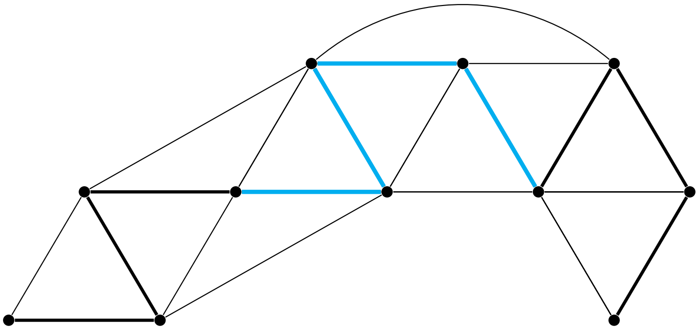
Root path and inverse root path

- Let G be a 4-connected triangulation
- Let $\mathcal{S}$ be the set of all root paths and inverse root paths ($|\mathcal{S}| = 12(n-2)$)
- r switches root path to inverse root path on same vertices (and vice versa) ($\bar{\mathcal{S}} = \mathcal{S}$)
- $\mathcal{C}' = \mathcal{C}$

This gives a counting base $(\mathcal{S}, r)$ for $\mathcal{C}$.

Theorem (Brinkmann, Souffriau, NVC, 2014)

For each 4-connected triangulation G we have that $O_{\mathcal{S}}(\mathcal{C}, r) \leq 5$ with $\mathcal{S}$ the set of all root and inverse root paths in G.



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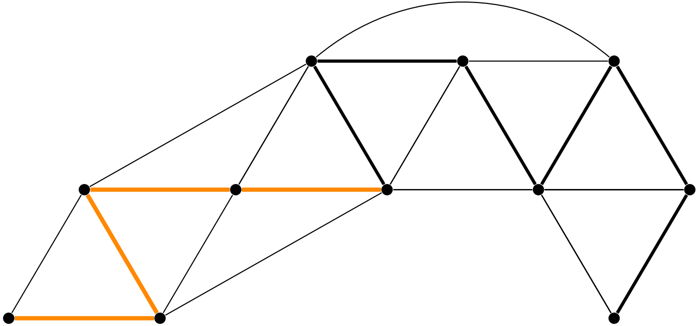

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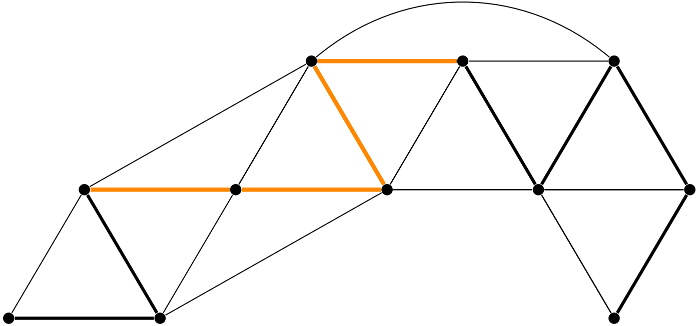

Theorem (Brinkmann, Souffriau, NVC, 2014)

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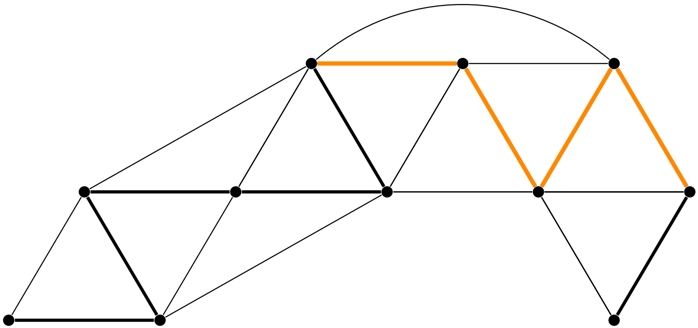

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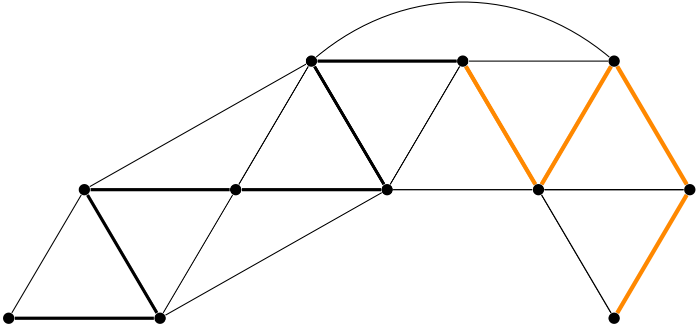


Theorem (Brinkmann, Souffriau, NVC, 2014)

Every 4-connected triangulation on n vertices has at least $\frac{12}{5}(n-2)$ *hamiltonian cycles.*

Maximum overlap of a subset $\mathcal{S}_i$ of $\mathcal{S}$

Let $(\mathcal{S}, r)$ be a counting base for $\mathcal{C}'$, and let $\mathcal{S}_i$ be a subset of $\mathcal{S}$:

$$O_{\mathcal{S}|\mathcal{S}_i}(\mathcal{C}', r) = \max\{o_{\mathcal{S}}(C, S) \mid C \in \mathcal{C}', S \in \mathcal{S}_i : S \subseteq C\}$$

Theorem (Brinkmann, Cuvelier, NVC, 2015)

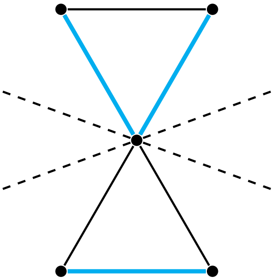
Given a graph G, a set $\mathcal{C}' \subseteq \mathcal{C}$, a nonempty counting base $(\mathcal{S}, r)$ for $\mathcal{C}'$, and a partition $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_k$ of $\mathcal{S}$, then

$$|\mathcal{C}'| \geq \sum_{i=1}^{k} |\mathcal{S}_i| \frac{2}{O_{\mathcal{S}}(\mathcal{C}', r) + O_{\mathcal{S}|\mathcal{S}_i}(\mathcal{C}', r)}.$$

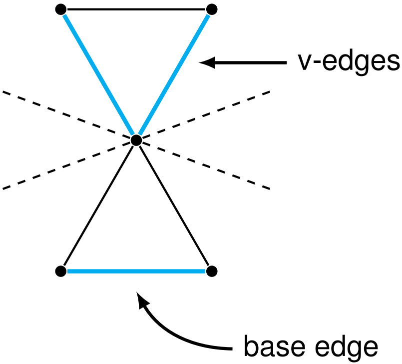
Theorem (Brinkmann, Cuvelier, NVC, 2015)

Every 4-connected triangulation on n vertices has at least $\frac{161}{60}(n-2)$ hamiltonian cycles.

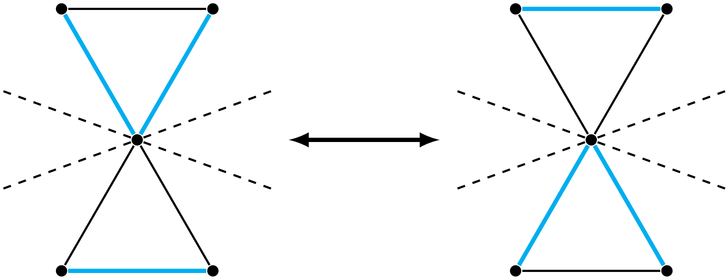
Hourglasses



Hourglasses



Inverse hourglass



Sparse set of hourglasses for a triangle T

Definition

A set $\mathcal{H}$ of hourglasses is a sparse set for a triangle T if

- no two elements $H \neq H'$ have the same set of v-edges, and
- for each $H \in \mathcal{H}$: if T is not one of the two triangles of H, then H contains no edge of T as v-edge or base edge.

Lemma

A 4-connected triangulation with n vertices contains a sparse set of hourglasses of size $6n - 21$ for each facial triangle.

Set of hamiltonian cycles sharing an edge with a face

$\mathcal{C}_T$: set of all hamiltonian cycles in graph G that share an edge with the face T.

- Let G be a 4-connected triangulation
- Let T be a face of G
- Let $\mathcal{H}$ be a sparse set of hourglasses for T
- r maps an hourglass to its inverse

This gives a counting base $(\mathcal{H}, r)$ for $\mathcal{C}_T$.

Set of hamiltonian cycles sharing an edge with a face

The overlap for hourglasses is at most 4, so…

$$\Rightarrow |\mathcal{C}_T| \geq \frac{6n-21}{O_{\mathcal{H}}(\mathcal{C}_T,r)} \geq \frac{6n-21}{4}$$

One separating triangle

Theorem (Brinkmann, Souffriau, NVC, 2014)

Every 3-connected triangulation on n vertices with exactly one separating triangle has at least $\frac{6n-27}{4}$ hamiltonian cycles.

Outline

Theorem (Brinkmann, Cuvelier, NVC, 2015)

Every 4-connected triangulation on n vertices has at least $\frac{161}{60}(n-2)$ hamiltonian cycles.

Theorem (Brinkmann, Souffriau, NVC, 2014)

Every 3-connected triangulation on n vertices with exactly one separating triangle has at least $\frac{6n-27}{4}$ hamiltonian cycles.

Computational results

Conjectured bound for 4-connected triangulations by Hakimi, Schmeichel and Thomassen verified **up to 25** vertices.

Computational results

For 6 separating triangles it is known that there exist non-hamiltonian 3-connected triangulations.

Minimum number of hamiltonian cycles for 3-connected triangulations with at most 5 separating triangles computed **up to 23** vertices.

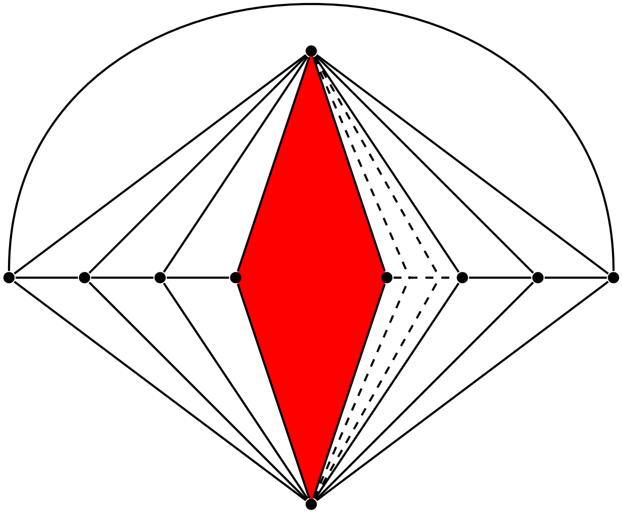
Computational results

	1	2	3	4	5
5	6	-	-	-	-
6	-	10	-	-	-
7	24	-	12	-	-
8	42	26	-	6	-
9	64	36	24	-	8
10	90	46	33	12	-
11	120	56	41	14	12
12	154	66	49	14	10
13	192	76	57	14	10
14	234	86	65	14	10
15	280	96	73	14	10
16	330	106	81	14	10
17	384	116	89	14	10
18	442	126	97	14	10
19	504	136	105	14	10
20	570	146	113	14	10
21	640	156	121	14	10
22	714	166	129	14	10
23	792	176	137	14	10

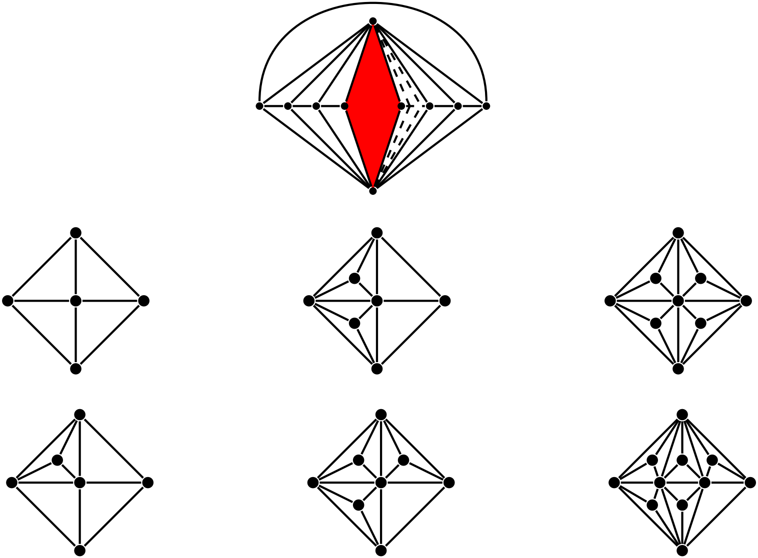
Computational results

	1	2	3	4	5
For $n \geq 12$	$2(n-1)(n-5)$	$10n-54$	$8n-47$	14	10

Extremal graphs



Extremal graphs



Summary

Lower bounds for the number of hamiltonian cycles in triangulations with few separating triangles

# sep. triangle	Old bound	New bound	Conjectured bound
0	$\frac{n}{\log_2 n}$	$\frac{161}{60}(n-2)$	$2(n-2)(n-4)$
1	4	$\frac{6n-27}{4}$	$2(n-1)(n-5)$
2	4	$[4, 10n-54]$	$10n-54$
3	4	$[4, 8n-47]$	$8n-47$
4	0	$[0, 14]$	14
5	0	$[0, 10]$	10

Outline

4-connected triangulations

- Better than constant bounds in case of two separating triangles
- Better than linear bounds in case of zero or one separating triangle

Other graphs

Counting base is not specific to triangulations, but no other examples are known!