Hamiltonian Cycles in Triangulations

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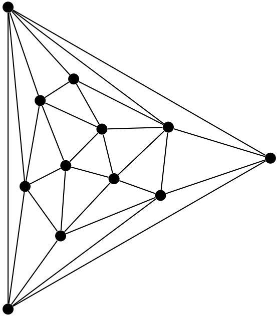
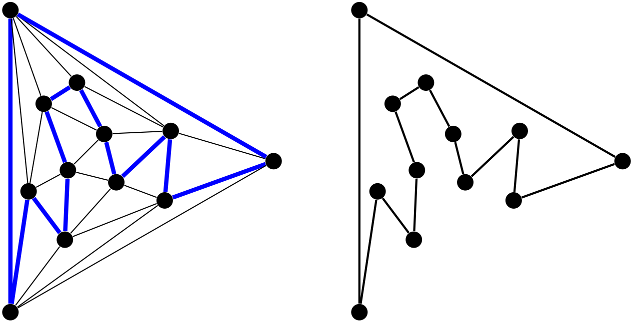
Triangulation

A triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

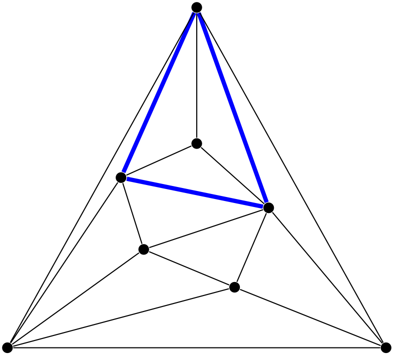
A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.



A graph is hamiltonian if it contains a hamiltonian cycle.

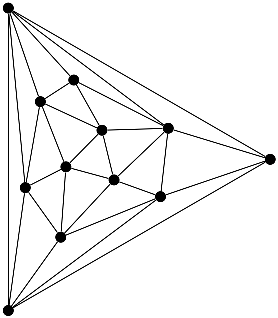
Separating triangles

A separating triangle S in a triangulation T is a subgraph of T such that S is isomorphic to C_3 and $T - S$ has two components.



4-connected triangulations

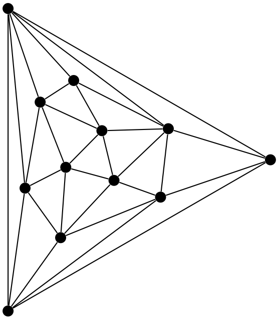
A triangulation is 4-connected if and only if it contains no separating triangles.



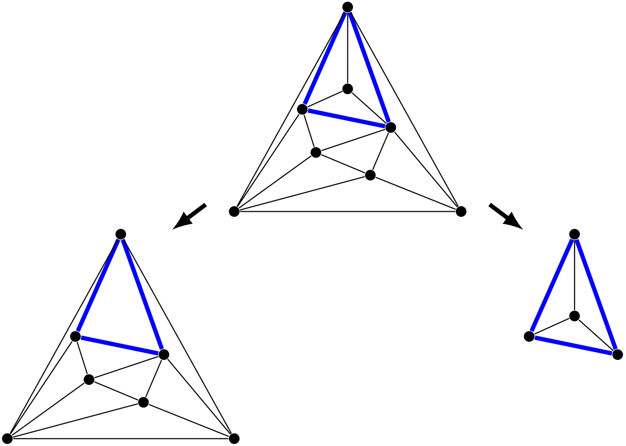
Whitney

Theorem (Whitney, 1931)

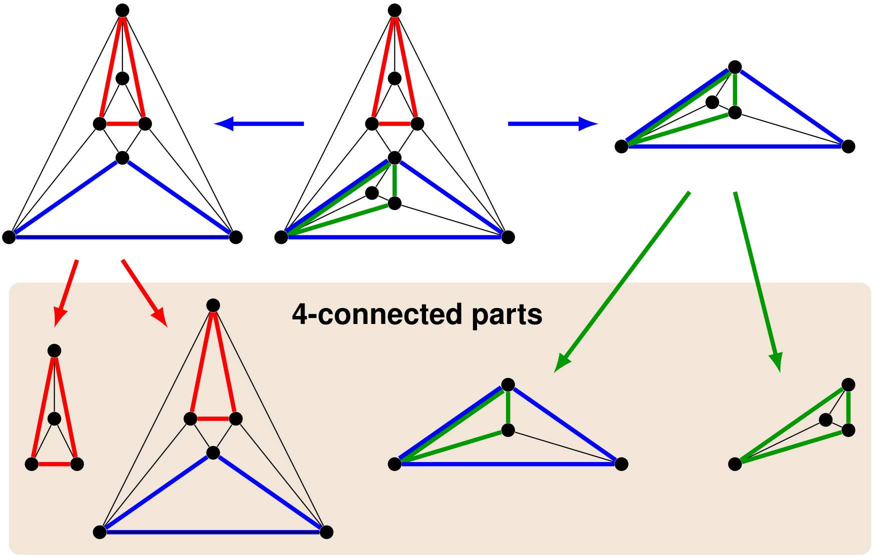
Each triangulation without separating triangles is hamiltonian.



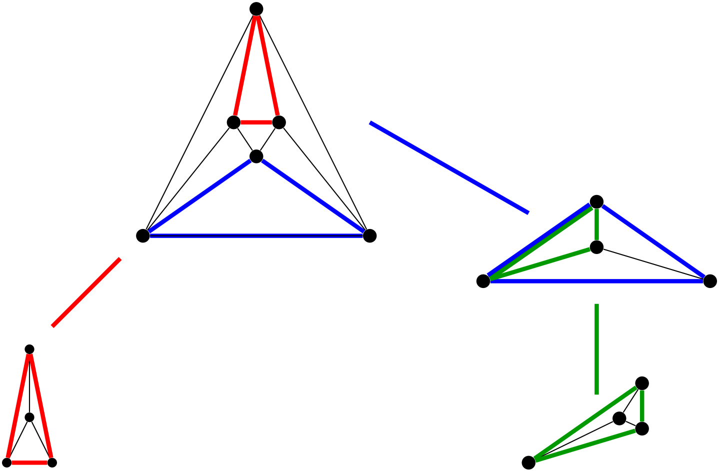
Splitting triangulations



Recursively splitting triangulations



Vertices: 4-connected parts
Edges: separating triangles

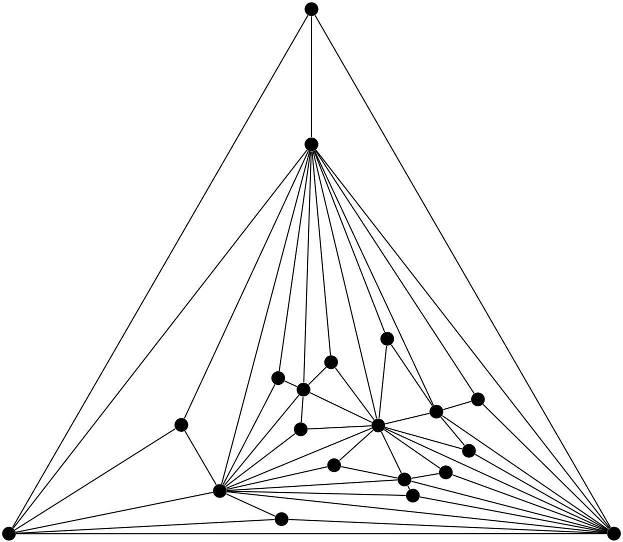


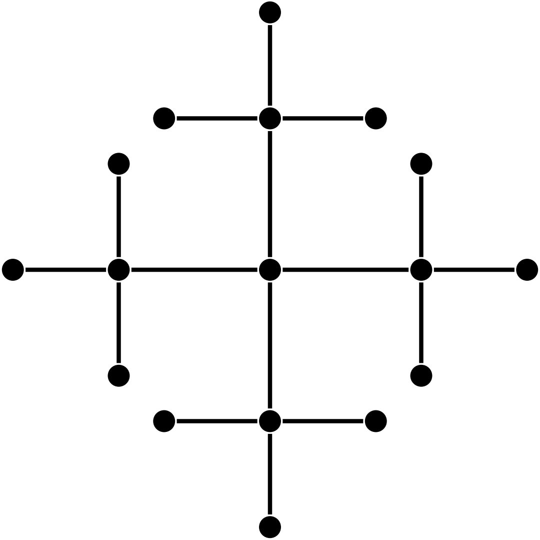
Jackson and Yu

Theorem (Jackson and Yu, 2002)

A triangulation with a decomposition tree with maximum degree 3 is hamiltonian.

There exists a non-hamiltonian triangulation with a decomposition tree with maximum degree 4.





Question

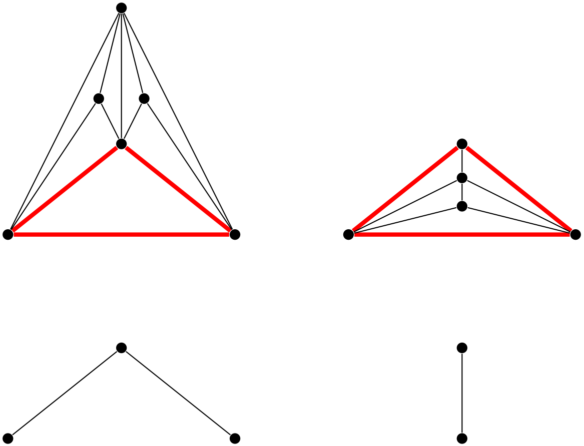
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?

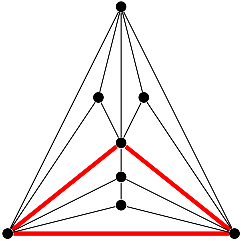
Theorem (Jackson and Yu, 2002)

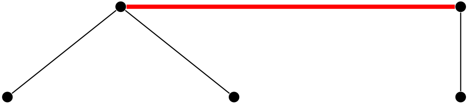
Let G be a 4-connected triangulation. Let T, T_1, T_2 be distinct triangles in G. Let $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$ and $e_2 \in E(T_2)$ such that uv, uw, e_1 and e_2 are distinct and contained in $E(C)$.

Subdividing a face with a graph



Subdividing a face with a graph

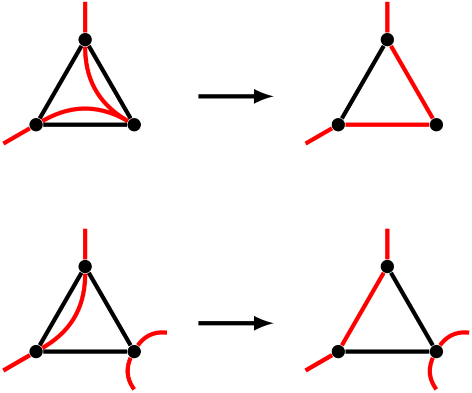




Subdividing a non-hamiltonian triangulation

Lemma

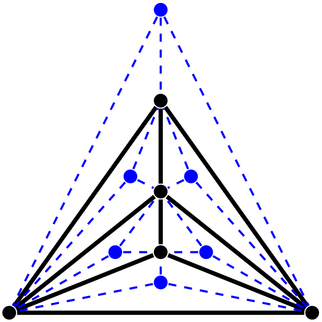
When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.



Creating a non-hamiltonian plane graph

Lemma

When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.



The subdivided graph is not 1-tough.

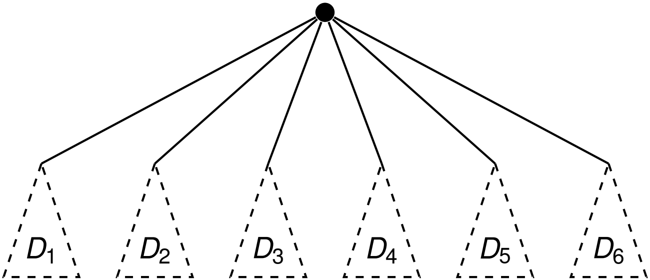
Decomposition trees with $\Delta \geq 6$

Theorem

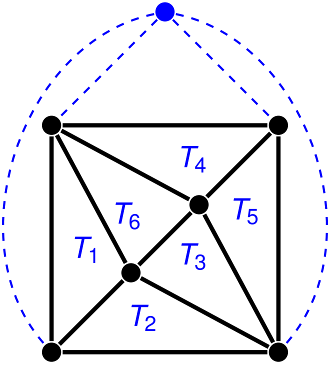
For each tree D with $\Delta(D) \geq 6$, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.

Constructive proof.

Assume $\Delta(D) = 6$.

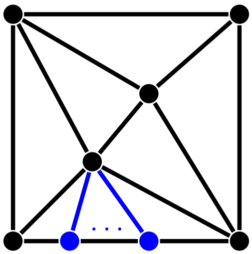


Choose triangulation T_i with decomposition tree D_i $(1 \leq i \leq 6)$



A non-hamiltonian triangulation with D as decomposition tree.

$\Delta(D) > 6$



Remaining cases

Given a tree D:

If $\Delta(D) \leq 3$, then D is **not** the decomposition tree of a non-hamiltonian triangulation.

If $\Delta(D) \geq 6$, then D is the decomposition tree of a non-hamiltonian triangulation.

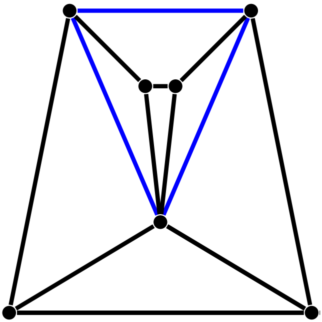
What if $\Delta(D) = 4$ or $\Delta(D) = 5$?

Multiple degrees > 3

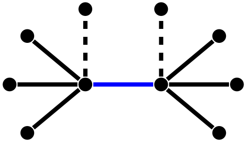
Theorem

For each tree D with at least two vertices with degree > 3, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.

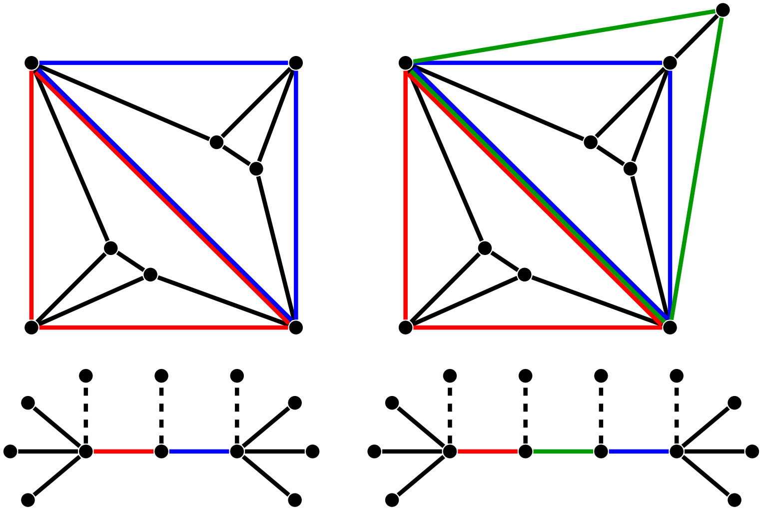
Adjacent vertices with degree > 3



8 faces and 7 vertices



Non-adjacent vertices with degree > 3



Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.

Simplifying things

Theorem

For each $k \geq 4$. Let D be a tree with one vertex of degree k and all other vertices of degree ≤ 3.
There exists a non-hamiltonian triangulation with D as decomposition tree if and only if there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree.

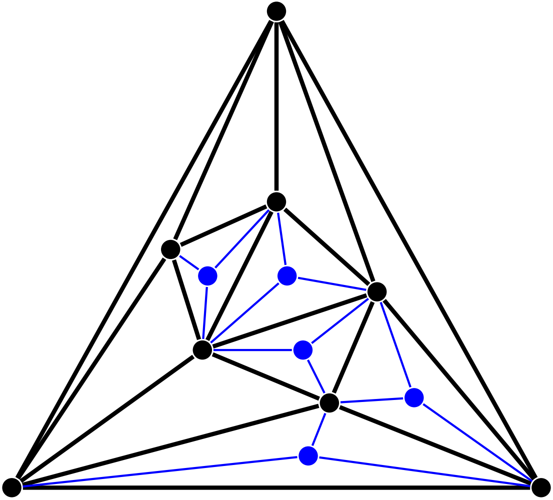
Simplifying things (more)

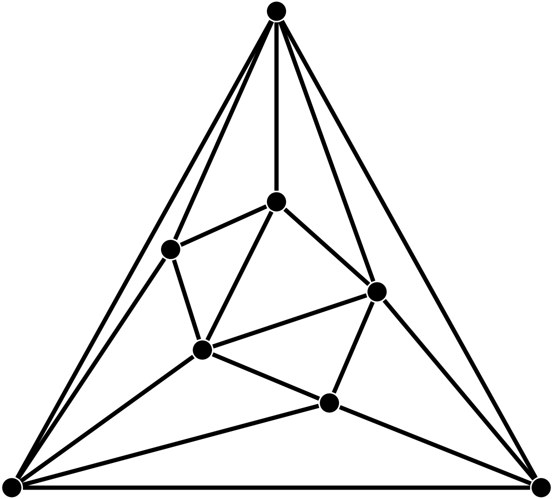
Theorem

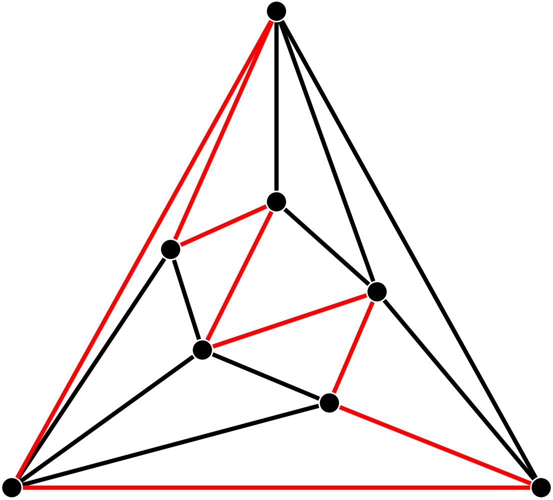
For each $k \geq 4$. If there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree, then there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree such that the leaves correspond to K_4's.

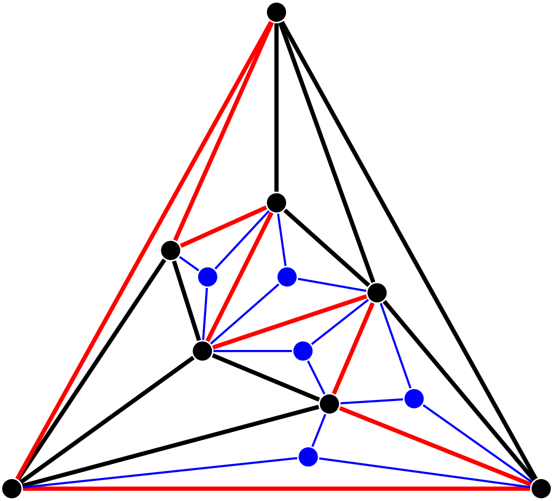
Specialized search

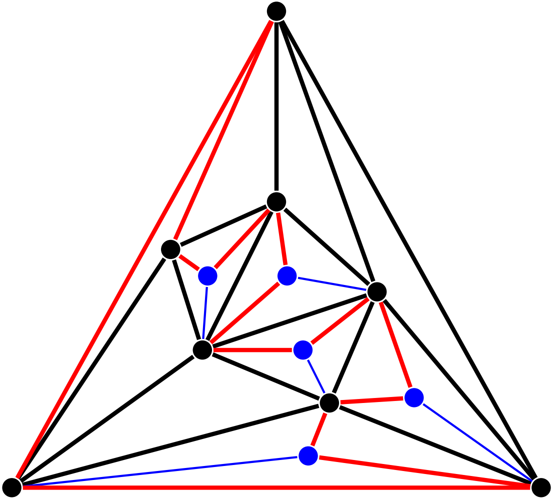
Specialised programs to search for non-hamiltonian triangulations with $K_{1,4}$ or $K_{1,5}$ as decomposition tree.









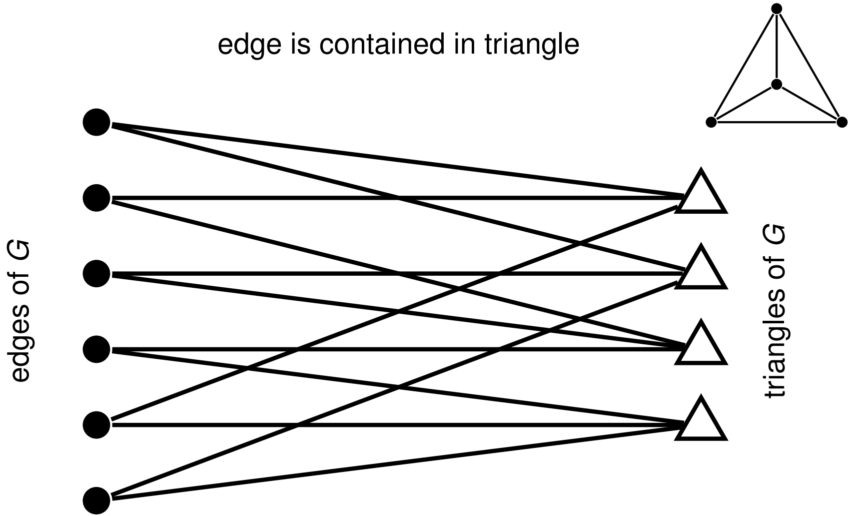


Extending hamiltonian cycles

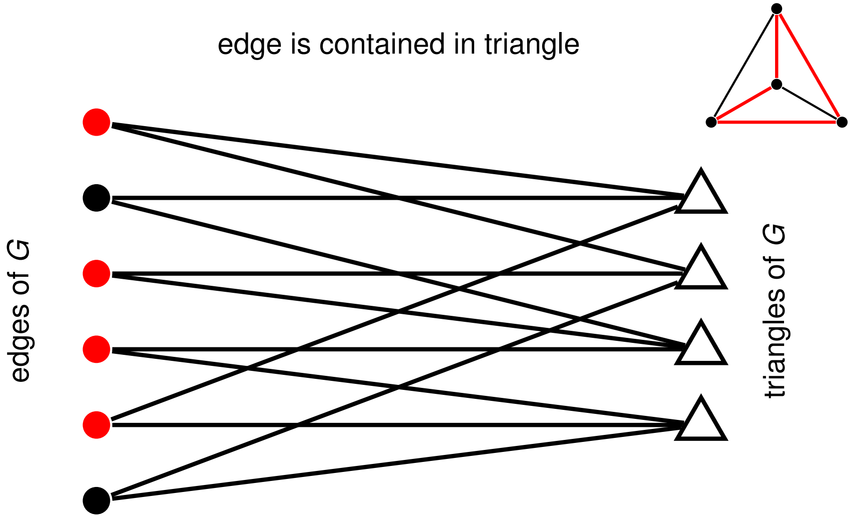
Given a graph G and the graph G' which is constructed from G by subdividing 4 or 5 faces with a K_4.

When can a hamiltonian cycle of G be extended to a hamiltonian cycle of G'?

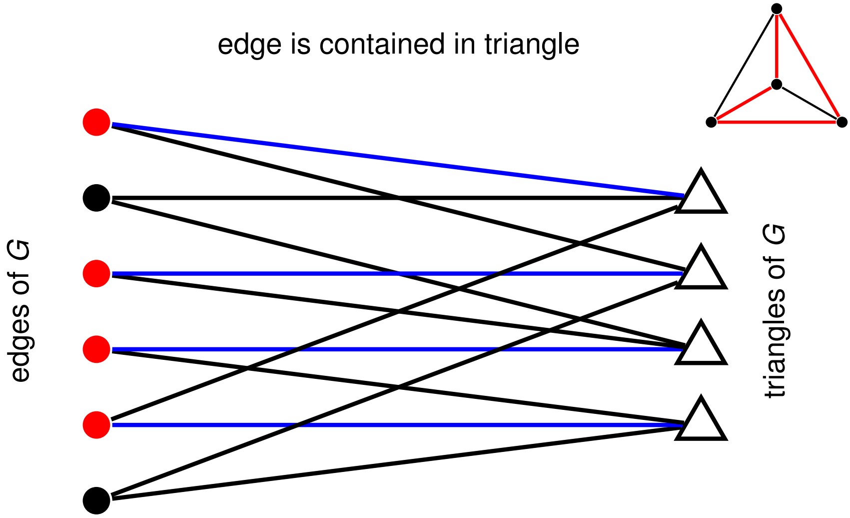
Hamiltonian cycles and matchings



Hamiltonian cycles and matchings



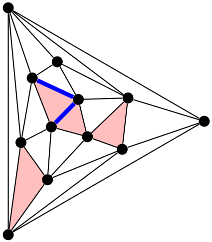
Hamiltonian cycles and matchings



Limiting the 4-tuples

Theorem

Let G be a 4-connected triangulation. Let T_1, T_2, T_3 and T_4 be triangles in G such that at least two of them share an edge. The graph obtained by subdividing the four triangles with a K_4 is hamiltonian.



$\Rightarrow$ only check edge-disjoint 4-tuples of faces

Hitting each triangle

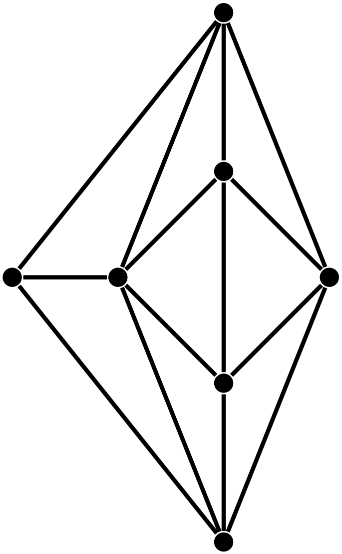
Wish

If there would always be a Hamiltonian cycle that shares an edge with each triangle, then this would be solved.

Theorem

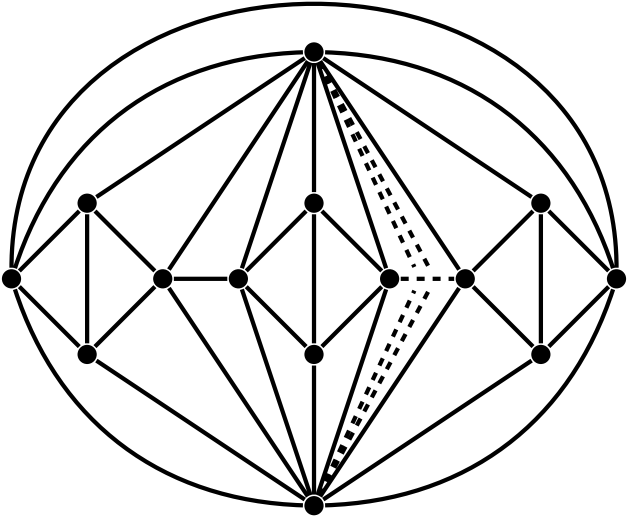
For each $k > 1$, there exists a 4-connected triangulation T such that for each hamiltonian cycle C in T, there exist at least k faces that do not share an edge with C.

Missing triangles

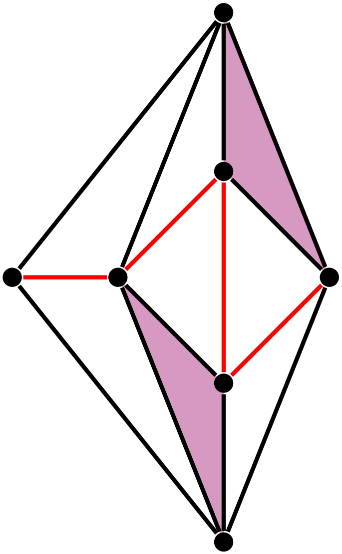


$4 + \lceil \frac{k}{2} \rceil$ copies

Missing triangles



Missing triangles



at least $\lceil \frac{k}{2} \rceil$ times

Results

All triangulations on at most 27 vertices with $K_{1,4}$ or $K_{1,5}$ as decomposition tree are hamiltonian.

Results

V	F	4-connected triangulations
6	8	1
7	10	1
8	12	2
9	14	4
10	16	10
11	18	25
12	20	87
13	22	313
14	24	1357
15	26	30 926
16	28	158 428
17	30	836 749
18	32	4 504 607
19	34	24 649 284
20	36	136 610 879
21	38	765 598 927
22	40	4 332 047 595
23	42	24 724 362 117

... and now?

Prove that for each 4-tuple of edge-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.

... and now?

Prove that for each 5-tuple of triangles T_1, T_2, T_3, T_4, T_5 in a 4-connected triangulation there exists a hamiltonian cycle C and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

Find a counterexample.

Thanks for your attention.