Non-Hamiltonian and Non-Traceable Regular 3-Connected Planar Graphs

Nico Van Cleemput Carol T. Zamfirescu

Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics, Computer Science and Statistics
Ghent University

GHENT UNIVERSITY

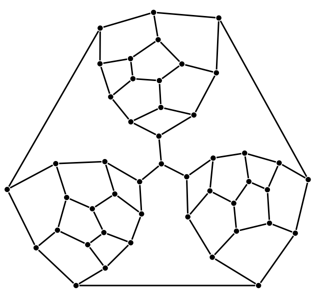
- Here, a *polyhedron* is a planar 3-connected graph.
- The word “regular” is used exclusively in the graph-theoretical sense of having all vertices of the same degree.
- By Euler’s formula, there are k-regular polyhedra for exactly three values of k: 3, 4, or 5.

- Let c_k be the order of the smallest **non-hamiltonian** k-regular polyhedron.
- Let p_k be the order of the smallest **non-traceable** k-regular polyhedron.

Cubic polyhedra – hamiltonicity

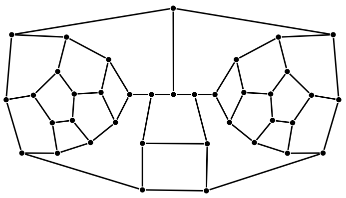
- Tait conjectured in 1884 that every cubic polyhedron is hamiltonian.
- The conjecture became famous because it implied the Four Colour Theorem (at that time still the Four Colour Problem)

Cubic polyhedra – hamiltonicity



The first to construct a counterexample (of order 46) was Tutte in 1946

Cubic polyhedra – hamiltonicity



Lederberg, Bosák, and Barnette (pairwise independently) described a smaller counterexample having 38 vertices.

Cubic polyhedra – hamiltonicity

After a long series of papers by various authors (e.g., Butler, Barnette, Wegner, Okamura), Holton and McKay showed that all cubic polyhedra on up to 36 vertices are hamiltonian.

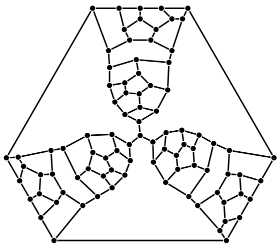
Theorem (Holton and McKay, 1988)

$c_3 = 38$

Cubic polyhedra – traceability

- Balinski asked whether cubic non-traceable polyhedra exist
- Brown and independently Grünbaum and Motzkin proved the existence of such graphs
- Klee asked for determining p_3

Cubic polyhedra – traceability



In 1970 T. Zamfirescu constructed this cubic non-traceable planar graph on 88 vertices

Cubic polyhedra – traceability

Based on work of Okamura, Knorr improved a result of Hoffmann by showing that all cubic planar graphs on up to 52 vertices are traceable.

Theorem (Knorr, 2010 and Zamfirescu, 1970)

$54 \leq p_3 \leq 88$

Quartic polyhedra – hamiltonicity

- Following work of Sachs from 1967 and Walther from 1969, Zaks proved in 1976 that there exists a quartic non-hamiltonian polyhedron of order 209.
- The actual number given in Zaks' paper is false, as pointed out in work of Owens — therein the correct number can be found.

Quartic polyhedra – hamiltonicity

Theorem (Sachs, 1967)

If there exists a non-hamiltonian (non-traceable) cubic polyhedron of order n, then there exists a non-traceable (non-hamiltonian) quartic polyhedron on $\frac{9n}{2}$ vertices.

On page 132 of Bosák's book it is claimed that converting the Lederberg-Bosák-Barnette graph with this method gives a quartic non-hamiltonian polyhedron of order 161. However, the correct number should be $38 \times \frac{9}{2} = 171$.

Theorem (Sachs, 1967 combined with Bosák, 1990)

$c_4 \leq 171$

Quartic polyhedra – traceability

- Zaks showed that $p_4 \leq 484$
- Using Sachs' theorem on Zamfirescu's 88-vertex graph gives a non-traceable quartic polyhedron on 396 vertices.

Theorem (Sachs, 1967 combined with Zamfirescu, 1970)

$p_4 \leq 396$

Quintic polyhedra

- Previous work includes papers by Walther, as well as Harant, Owens, Tkáč, and Walther.
- Zaks showed that $c_5 \leq 532$ and $p_5 \leq 1232$.
- Owens proved that $c_5 \leq 76$ and $p_5 \leq 128$.

Theorem (Owens, 1980)

$c_5 \leq 76$
$p_5 \leq 128$

Summary

	Hamiltonicity	Traceability
Cubic	$c_3 = 38$	$54 \leq p_3 \leq 88$
Quartic	$c_4 \leq 171$	$p_4 \leq 396$
Quintic	$c_5 \leq 76$	$p_5 \leq 128$

Cubic Polyhedra

Essentially 4-connected cubic polyhedra

Theorem (Aldred, Bau, Holton, and McKay, 2000)

Every essentially 4-connected cubic planar graph of order at most 40 is hamiltonian. Furthermore, there exist non-hamiltonian examples of order 42.

Theorem (Van Cleemput and Zamfirescu, 2018)

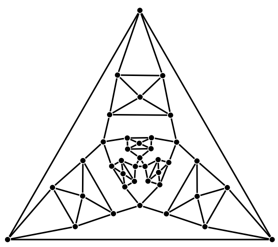
There exists a non-hamiltonian essentially 4-connected cubic polyhedron of order n if and only if n is even and $n \geq 42$.

Quartic Polyhedra

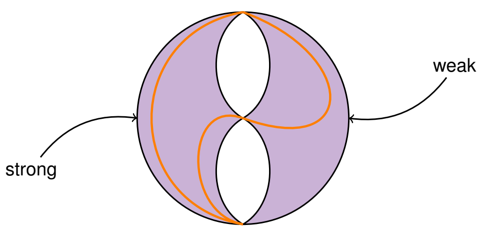
Upper bound hamiltonicity

Theorem (Van Cleemput and Zamfirescu, 2018)

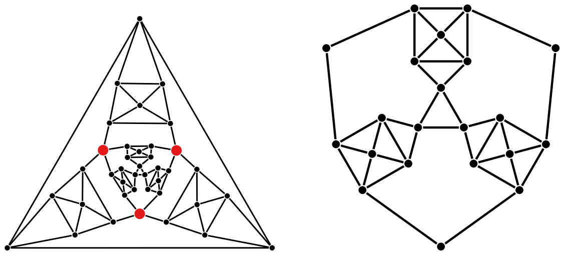
$c_4 \leq 39$



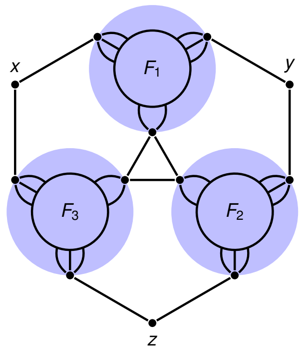
Upper bound hamiltonicity



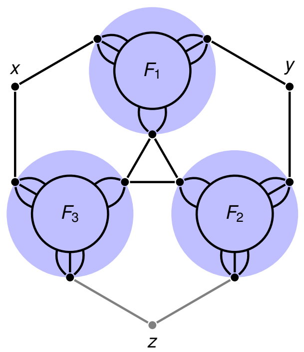
Upper bound hamiltonicity



Upper bound hamiltonicity



Upper bound hamiltonicity

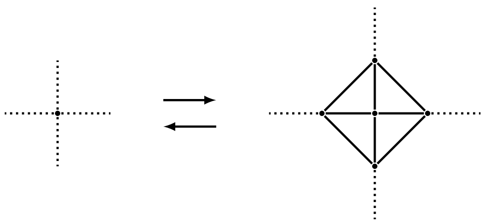


Lower bound hamiltonicity

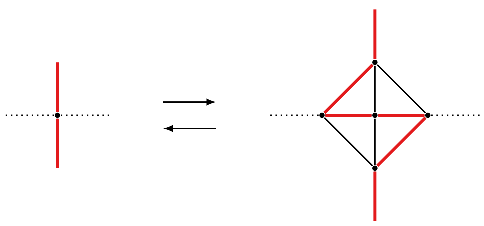
Check all quartic polyhedra for being hamiltonian.

Simple backtracking algorithm that tries to construct a cycle from some vertex.

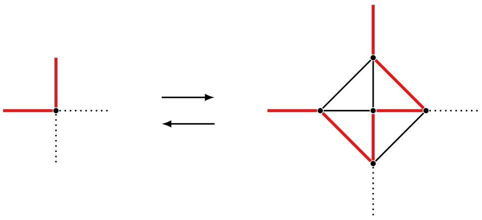
Lower bound hamiltonicity



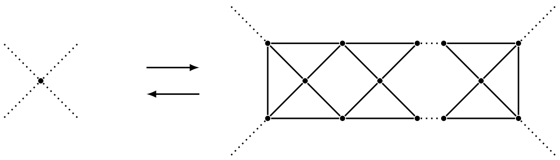
Lower bound hamiltonicity



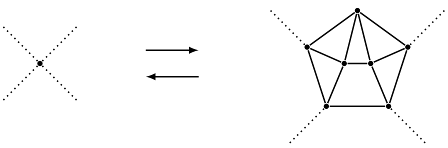
Lower bound hamiltonicity



Lower bound hamiltonicity



Lower bound hamiltonicity



Lower bound hamiltonicity

Theorem (Van Cleemput and Zamfirescu, 2018)

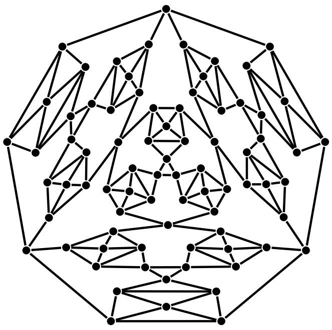
$c_4 \geq 35$

Vertices	Time
25	9.6 minutes
26	42.1 minutes
27	3.2 hours
28	15.1 hours
29	3.1 days
30	15.3 days
31	78.2 days
32	1.1 years
33	5.9 years
34	37.9 years

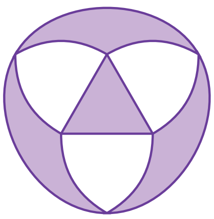
Upper bound traceability

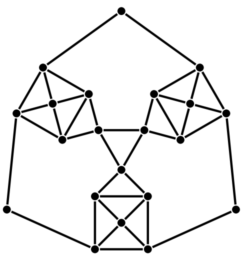
Theorem (Van Cleemput and Zamfirescu, 2018)

$p_4 \leq 78$



Upper bound traceability





$21 \times 4 - 6 = 78$ vertices

Lower bound traceability

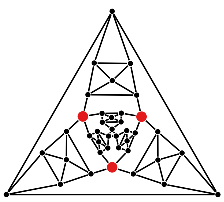
Lemma (Van Cleemput and Zamfirescu, 2018)

$p_4 \geq c_4 + 1$

Theorem (Van Cleemput and Zamfirescu, 2018)

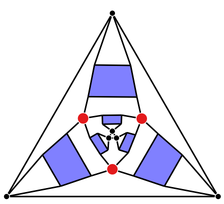
$p_4 \geq 36$

Further properties



- Not *homogeneously traceable*
- *Circumference* is $5 \times 6 + 4 = 34$.

Further properties



- For each $n \geq 39$ there is a quartic polyhedron on n vertices that is not homogeneously traceable.
- For the family $\mathcal{G}$ of quartic polyhedra, the *shortness coefficient* $\rho(\mathcal{G})$ is at most $\frac{5}{6}$:

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\operatorname{circ}(G)}{|V(G)|} \leq \liminf_{k \to \infty} \frac{5k+4}{6k+3} = \frac{5}{6}$$

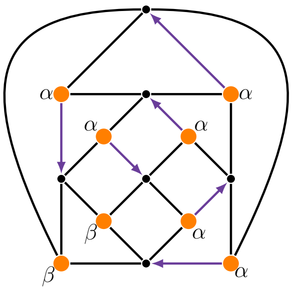
.

Quintic Polyhedra

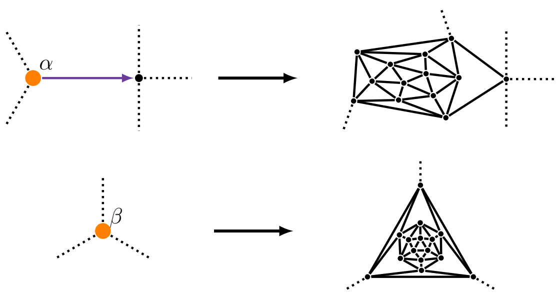
Upper bound traceability

Theorem (Van Cleemput and Zamfirescu, 2018)

$p_5 \leq 108$



Upper bound traceability



$6 \times 13 + 2 \times 15 = 108$ vertices

Upper bound traceability

Theorem (Van Cleemput and Zamfirescu, 2018)

There exists a quintic non-traceable polyhedron of order n for every even $n \geq 108$.

Summary

	Hamiltonicity	Traceability
Cubic	$c_3 = 38$	$54 \leq p_3 \leq 88$
Quartic	$\cancel{c_4 \leq 171}$ $35 \leq c_4 \leq 39$	$\cancel{p_4 \leq 396}$ $36 \leq p_4 \leq 78$
Quintic	$\cancel{c_5 \leq 76}$ $38 \leq c_5 \leq 76$	$\cancel{p_5 \leq 128}$ $38 \leq p_5 \leq 108$

Future work

- Increase lower bound for c_4
  - number of 3-cuts
  - required subgraphs
- Lower bounds for quintic case
- Lower bounds for traceability
- Upper bound for hamiltonicity of quintic case