On the strongest form of a theorem of Whitney for
hamiltonian cycles in plane triangulations

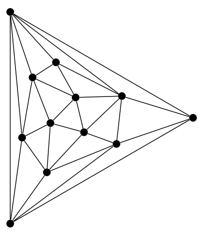
Gunnar Brinkmann[1] Jasper Souffriau[1]
Nico Van Cleemput[2]

[1]Ghent University

[2]University of West Bohemia

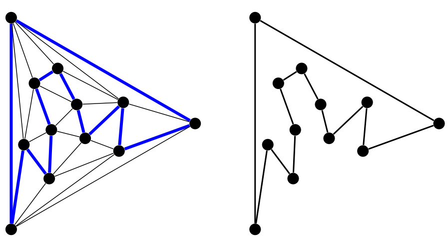
Triangulation

A triangulation is a plane graph in which each face is a triangle.



Hamiltonian cycle

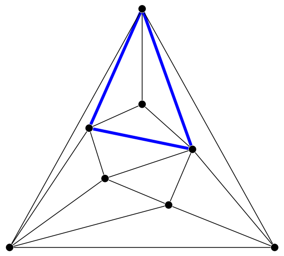
A hamiltonian cycle in $G(V, E)$ is a subgraph of $G(V, E)$ which is isomorphic to $C_{|V|}$.



A graph is hamiltonian if it contains a hamiltonian cycle.

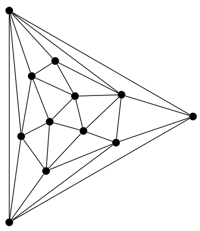
Separating triangles

A separating triangle S in a triangulation T is a subgraph of T such that S is isomorphic to C_3 and $T - S$ has two components.



4-connected triangulations

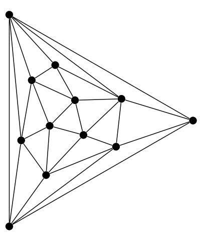
A triangulation is 4-connected if and only if it contains no separating triangles.



Whitney

Theorem (Whitney, 1931)

Each triangulation without separating triangles is hamiltonian.



Chen

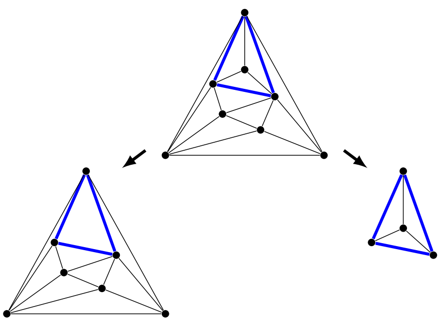
Theorem (Chen, 2003)

Each triangulation with at most one separating triangle is hamiltonian.

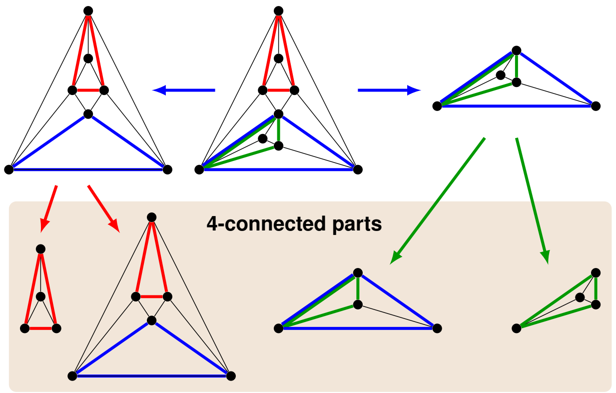
Jackson and Yu

The ‘next’ step (2002) is a strong improvement, but takes longer to explain.

Splitting triangulations

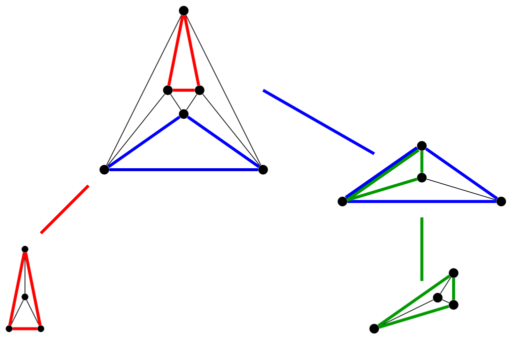


Recursively splitting triangulations



Decomposition tree

Vertices: 4-connected parts
Edges: separating triangles



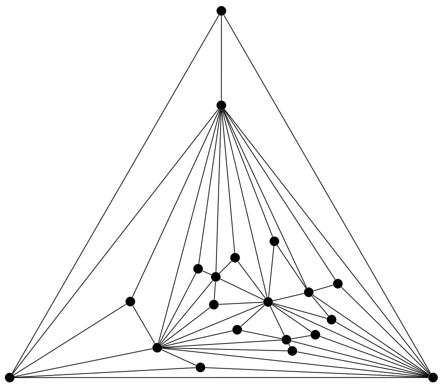
Jackson and Yu

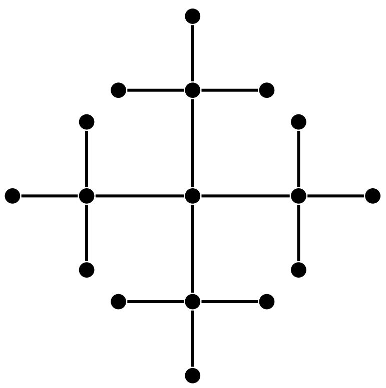
Theorem (Jackson and Yu, 2002)

A triangulation with a decomposition tree with maximum degree 3 is hamiltonian.

Jackson and Yu

There exists a non-hamiltonian triangulation with a decomposition tree with maximum degree 4.



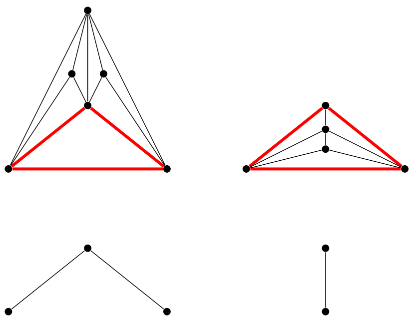


Question

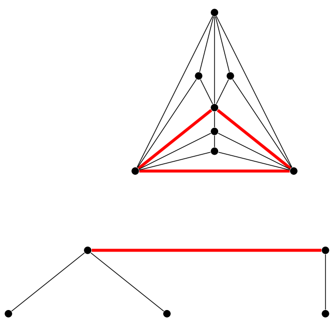
Can the result of Jackson and Yu be improved?

Which trees can arise as decomposition trees of non-hamiltonian triangulations?

Subdividing a face with a graph



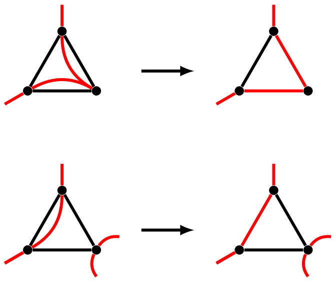
Subdividing a face with a graph



Subdividing a non-hamiltonian triangulation

Lemma

When a non-hamiltonian triangulation is subdivided, then the resulting graph is also non-hamiltonian.

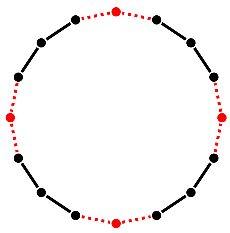


Toughness

A graph is 1-tough if it cannot be split into k components by removing less than k vertices.

Toughness

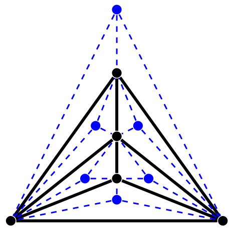
A hamiltonian graph is 1-tough.



Creating a non-hamiltonian plane graph

Lemma

When in a plane graph with more faces than vertices each face is subdivided, then the resulting plane graph is non-hamiltonian.



The subdivided graph is not 1-tough.

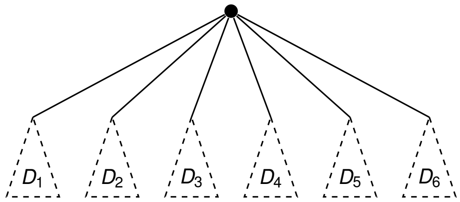
Decomposition trees with $\Delta \geq 6$

Theorem

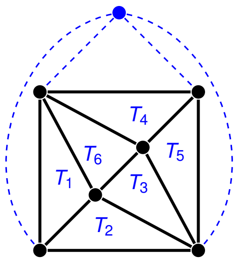
For each tree D with $\Delta(D) \geq 6$, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.

Constructive proof.

Assume $\Delta(D) = 6$.

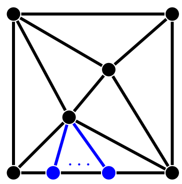


Choose triangulation T_i with decomposition tree D_i ($1 \leq i \leq 6$)



A non-hamiltonian triangulation with D as decomposition tree.

$\Delta(D) > 6$



Remaining cases

Given a tree D:

If $\Delta(D) \leq 3$, then D is **not** the decomposition tree of a non-hamiltonian triangulation.

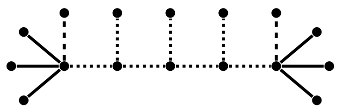
If $\Delta(D) \geq 6$, then D is the decomposition tree of a non-hamiltonian triangulation.

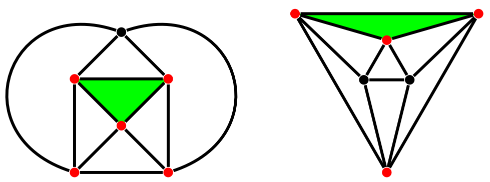
What if $\Delta(D) = 4$ or $\Delta(D) = 5$?

Multiple degrees > 3

Theorem

For each tree D with at least two vertices with degree > 3, there exists a non-hamiltonian triangulation T, such that D is the decomposition tree of T.





red vertices: $5 + (k - 1) + (5 - 3) = 6 + k$
components: $4 + (k - 1) + 4 = 7 + k$

Remaining cases: trees with one vertex of degree 4 or 5 and all other degrees at most 3.

One vertex of degree 4 or 5

Theorem

Let T be a triangulation with decomposition tree D with only one vertex of degree 4 or 5 and all other vertices of degree at most 3. Then T is 1-tough.

Simplifying things

Theorem

For each $k \geq 4$. Let D be a tree with one vertex of degree k and all other vertices of degree ≤ 3.
There exists a non-hamiltonian triangulation with D as decomposition tree if and only if there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree.

Simplifying things (more)

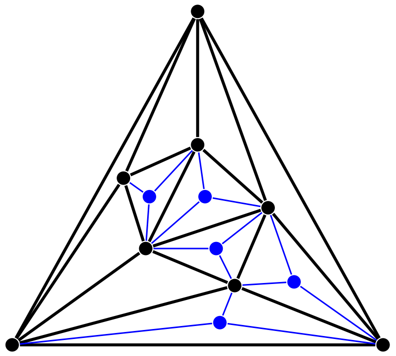
Theorem

For each $k \geq 4$. If there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree, then there exists a non-hamiltonian triangulation with $K_{1,k}$ as decomposition tree such that the leaves correspond to K_4's.

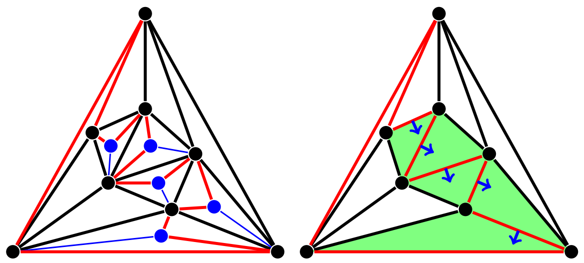
Specialised search

Specialised programs to search for non-hamiltonian triangulations with $K_{1,4}$ or $K_{1,5}$ as decomposition tree.

Smallest non-hamiltonian triangulation with tree $K_{1,k}$



Smallest non-hamiltonian triangulation with tree $K_{1,k}$



Algorithm

- Find all k-tuples of faces
- Look for a hamiltonian cycle
  - Remove all *saturated* tuples
- while there are still tuples: look for new hamiltonian cycle

Jackson and Yu

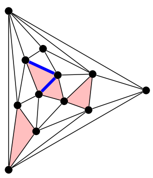
Theorem

Let G be a 4-connected triangulation. Let T, T_1, and T_2 be distinct triangles in G. Let $V(T) = \{u, v, w\}$. Then there exists a hamiltonian cycle C of G and edges $e_1 \in E(T_1)$, $e_2 \in E(T_2)$, such that uv, uw, e_1, and e_2 are distinct and contained in C.

Limiting the 4-tuples

Theorem

Let G be a 4-connected triangulation. Let T_1, T_2, T_3, and T_4 be triangles in G such that at least two of them share an edge. The graph obtained by subdividing the four triangles with a K_4 is hamiltonian.



$\Rightarrow$ only check edge-disjoint 4-tuples of faces

Extended outerplanar discs

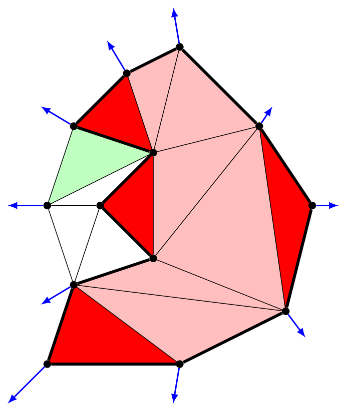
Outerplanar disc of a triangulation T
vertex induced subgraph of T which is an outerplanar triangulation of the disc with at least 3 vertices

Leaf of an outerplanar disc O
vertex which only has two neighbours in O, together with those two neighbours

Extended outerplanar disc of a triangulation T
outerplanar disc of T together with a triangle t not belonging to O, but sharing an edge with O

Leaf of an extended outerplanar disc O
Leaf of the outerplanar disc which contains a vertex of degree 2 in O which does not belong to t

Extended outerplanar discs



Extended outerplanar discs

Theorem

Let T be a 4-connected triangulation. Let $t_1, t_2, \ldots t_k$ (with $k \geq 4$) be distinct triangles in T such that there is an extended outerplanar disc O with extension t_1 containing $t_2, \ldots, t_{k-2}$ as leaves and with t_{k-1}, t_k not in O.
Then there exists a hamiltonian cycle C of T and edges $e_1 \in E(t_1), \ldots, e_k \in E(t_k)$ that are pairwise distinct and contained in $E(C)$.

Extended outerplanar discs

For each 4-tuple of faces in a plane triangulation:

There exists an eOPD such that one face is the extension, another face is a leaf and the other two faces are not contained in the eOPD.

⇕

There exists an eOPD containing at least two faces of the 4-tuple.

Extended outerplanar discs

For each 4-tuple of faces in a plane triangulation:

At least two faces of the 4-tuple share a vertex.

⇓

There exists an eOPD such that one face is the extension, another face is a leaf and the other two faces are not contained in the eOPD.

Extended outerplanar discs - algorithm

Extend tuple (called for each pair of faces)

- Check tuple
- If no eOPD was found:
  - Add face to tuple
  - Extend tuple

Check tuple

- Check the eOPD's in the list
- From each face: perform DFS-search
  - Greedily extend eOPD and store in list
  - Return
- Find hamiltonian cycle saturating tuple

Extended outerplanar discs - algorithm

- Initialise list with a good initial set of eOPD's
- Store OPD's together with set of extensions
- Faster to not check 2-tuples (often no eOPD)
- 3-tuples for which no eOPD exists are rare

Results

Two independent implementations of eOPD algorithm, both were ran for 4-connected triangulations on up to 27 vertices.

Fastest took 8.75 CPU years[1] for 27 vertices.

Two independent implementations of hamiltonian cycle algorithm, both were ran for 5-tuples in 4-connected triangulations on up to 22 vertices.

Fastest took:

- 4.1 months of CPU[1] for testing edge-disjoint 5-tuples
- 18.3 years of CPU[1] for testing not edge-disjoint 5-tuples

in all 4-connected triangulations on 22 vertices.

[1]Intel Xeon Nehalem (L5520) at 2.26 GHz

Results

All triangulations on at most 31 vertices with $K_{1,4}$ as decomposition tree are hamiltonian.

All triangulations on at most 27 vertices with $K_{1,5}$ as decomposition tree are hamiltonian.

Results

V	F	4-connected triangulations
6	8	1
7	10	1
8	12	2
9	14	4
10	16	10
11	18	25
12	20	87
13	22	313
14	24	1357
15	26	6244
16	28	30 926
17	30	158 428
18	32	836 749
19	34	4 504 607
20	36	24 649 284
21	38	136 610 879
22	40	765 598 927
23	42	4 332 047 595
24	44	24 724 362 117
25	46	142 205 424 580
26	48	823 687 567 019
27	50	4 801 749 063 379

... and now?

Prove that for each 4-tuple of vertex-disjoint triangles in a 4-connected triangulation there exists a hamiltonian cycle that shares an edge with each of the triangles.

or

Find a counterexample.

... and now?

Prove that for each 5-tuple of triangles T_1, T_2, T_3, T_4, T_5 in a 4-connected triangulation there exists a hamiltonian cycle C and distinct edges $e_1, e_2, e_3, e_4, e_5 \in C$ such that $e_i \in T_i$.

or

Find a counterexample.

Hitting each triangle

Wish

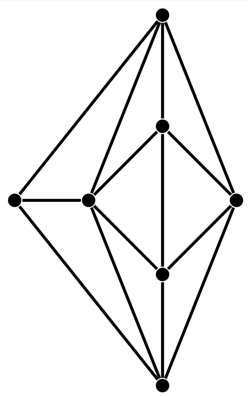
If there would always be a hamiltonian cycle that shares an edge with each triangle, then the 4-tuple case would be solved.

Missing triangles

Theorem

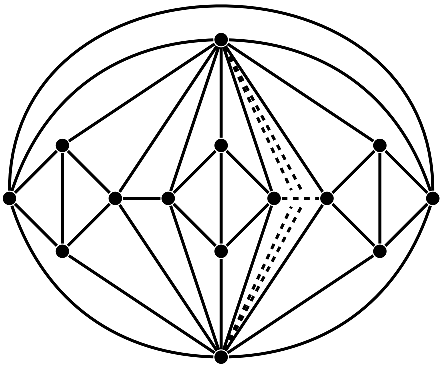
For each $k > 1$, there exists a 4-connected triangulation T such that for each hamiltonian cycle C in T, there exist at least k faces that do not share an edge with C.

Missing triangles

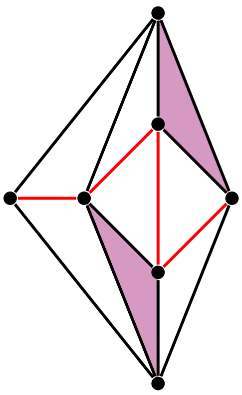


$4 + \lceil \frac{k}{2} \rceil$ copies

Missing triangles

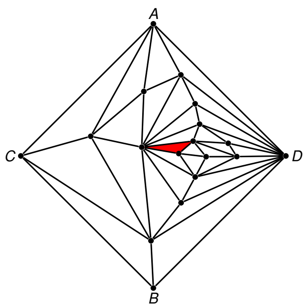


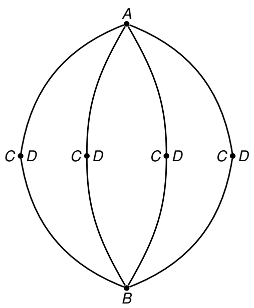
Missing triangles



at least $\lceil \frac{k}{2} \rceil$ times

Is there always an EOPD?





Oh, by the way...

Something on the number of hamiltonian cycles in triangulations with few separating triangles

Hakimi, Schmeichel and Thomassen (1979)
Each 4-connected triangulation contains at least $\frac{n}{\log_2 n}$ hamiltonian cycles.

Kratochvil and Zeps (1988)
Each hamiltonian triangulation contains at least 4 hamiltonian cycles.

Oh, by the way...

Lower bounds for the number of hamiltonian cycles in triangulations with few separating triangles

# sep. triangle	Previous bound	New bound	Conjectured bound
0	$\frac{n}{\log_2 n}$	$\frac{12n-24}{5}$	$2(n-2)(n-4)$
1	4	$\frac{6n-27}{4}$	$2(n-1)(n-5)$
2	4	$[4, 10n-54]$	$10n-54$
3	4	$[4, 8n-47]$	$8n-47$
4	0	$[0, 14]$	14?
5	0	$[0, 10]$	10?

Thanks for your attention.