4-connected polyhedra have at least a linear number of hamiltonian cycles

Gunnar Brinkmann Nico Van Cleemput

Combinatorial Algorithms and Algorithmic Graph Theory
Department of Applied Mathematics, Computer Science and Statistics
Ghent University

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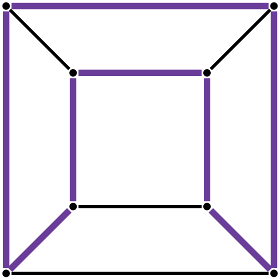
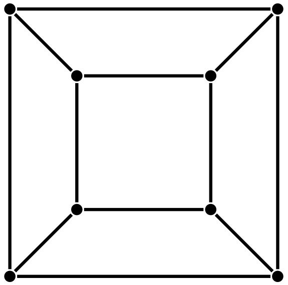
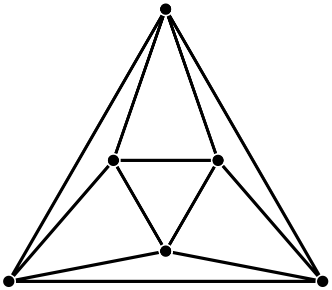
Hamiltonian cycles



A hamiltonian cycle is a spanning cycle.

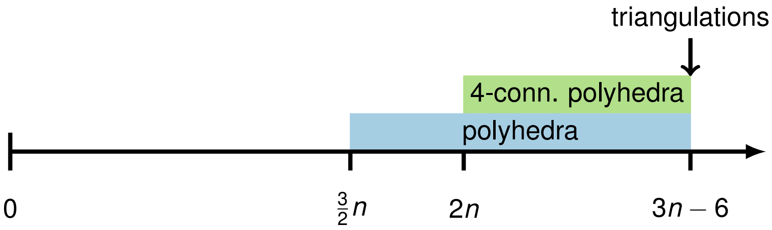
Polyhedra and triangulations





- Polyhedra are 3-connected plane graphs
- A triangulation is a polyhedron with only triangular faces

Edges in polyhedra on n vertices



More edges suggests: more likely to be hamiltonian!

90 years of theorems

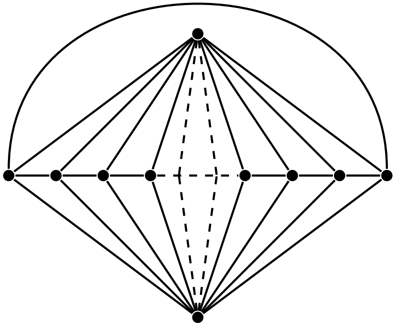
Triangulations		Polyhedra
4-conn. ⇒ hamiltonian Whitney (1931)	←25 years→	4-conn. ⇒ hamiltonian Tutte (1956)
at most three 3-cuts ⇒ hamiltonian Jackson, Yu (2002)	←17 years→	at most three 3-cuts ⇒ hamiltonian Brinkmann, Zamfirescu (2019)
six 3-cuts can be non-hamiltonian		six 3-cuts can be non-hamiltonian
four or five 3-cuts: unknown, but 1-tough		four or five 3-cuts: unknown, but 1-tough

Number of hamiltonian cycles

4-connected triangulations	4-connected polyhedra
≥ 1 hamiltonian cycle Whitney (1931)	≥ 1 hamiltonian cycle Tutte (1956)
$\geq \frac{n}{\log n}$ hamiltonian cycles Hakimi, Schmeichel, Thomassen (1979)	
	≥ 6 hamiltonian cycles Thomassen (1983)
$\geq \frac{12}{5}(n-2)$ hamiltonian cycles Brinkmann, Souffriau, VC (2018) $\geq \frac{161}{60}(n-2)$ hamiltonian cycles Brinkmann, Cuvelier, VC (2018)	

Number of hamiltonian cycles

- Up to 17 vertices there are 4-connected polyhedra with fewer hamiltonian cycles than the double wheel
- For 18 vertices or more the double wheel appears to be the polyhedron with the fewest number of hamiltonian cycles



$2(n-2)(n-4)$ hamiltonian cycles

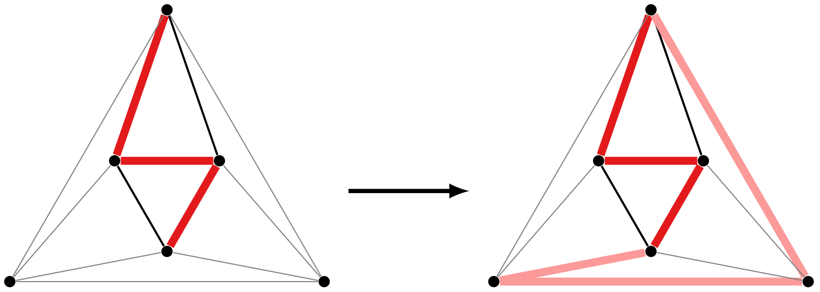
caagt

Hakimi, Schmeichel, Thomassen (1979)

Using a result of Whitney (1931):

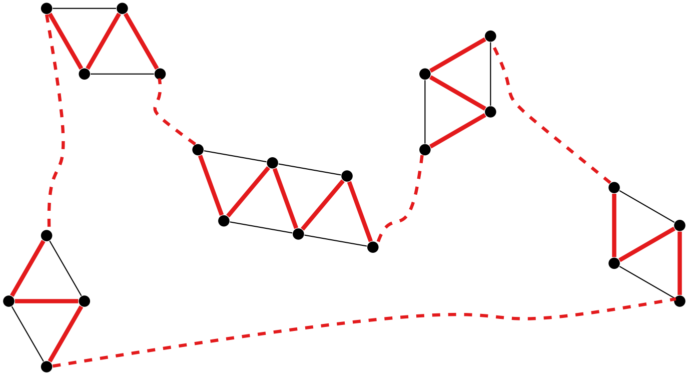
Lemma

Each zigzag in a 4-connected triangulation can be extended to a hamiltonian cycle.



Hakimi, Schmeichel, Thomassen (1979)

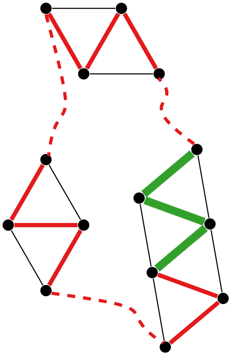
There is a linear number of such zigzags, but...

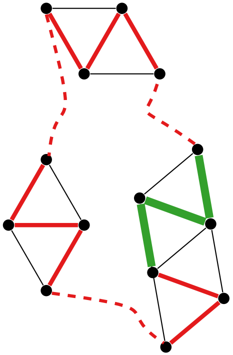


...a single hamiltonian cycle can contain a linear number of these zigzags, giving in total a constant number of hamiltonian cycles.

Hakimi, Schmeichel, Thomassen (1979)

A hamiltonian cycle with k disjoint zigzags guarantees 2^k hamiltonian cycles by '*switching*'.





This explains the $\frac{\cdot}{\log n}$ in the formula.

Counting bases

The main contribution of the 2018-paper:

counting differently via *counting bases*

Counting bases

Definition

Don't read this slide!

Let G be a [illegible]d let $\mathcal{C}$ be a collection of hamiltonian cycles of G. The pair $(\mathcal{S}, r)$, [illegible] $2^{E(G)}$ and r is a function $r : \mathcal{S} \rightarrow 2^{E(G)}$, is called a *counting b*[illegible]nd $\mathcal{C}$ if the pair $(\mathcal{S}, r)$ has the following properties:

(i) for all $S \in \mathcal{S}$, there is a ha[illegible]ycle $C \in \mathcal{C}$ saturating S.

(ii) for all $S \in \mathcal{S}$, $r(S) \subseteq E(G)$ (not [illegible] in $\mathcal{S}$) so that $S \not\subseteq r(S)$ and for each hamiltonian cycle $C \in$ [illegible] S we have that $z(C, S) = (C \setminus S) \cup r(S)$ is a hamiltonian [illegible]

(iii) for all $S_1 \neq S_2$, $S_1, S_2 \in \mathcal{S}$ and C saturating S_1 [illegible]$_2$, we have that $z(C, S_1) \neq z(C, S_2)$.

Counting bases

A *counting base* is a set of subgraphs (*switching subgraphs*) together with a function (*switching function*) satisfying 3 conditions:

(i) saturated

(ii) closed

(iii) independent

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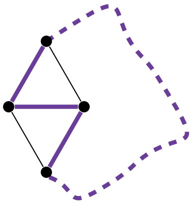

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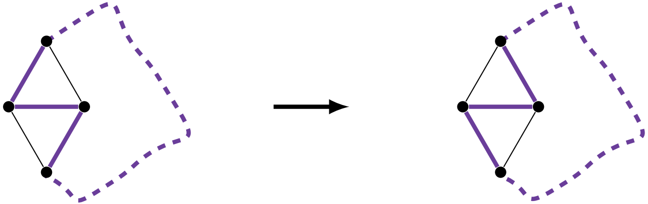

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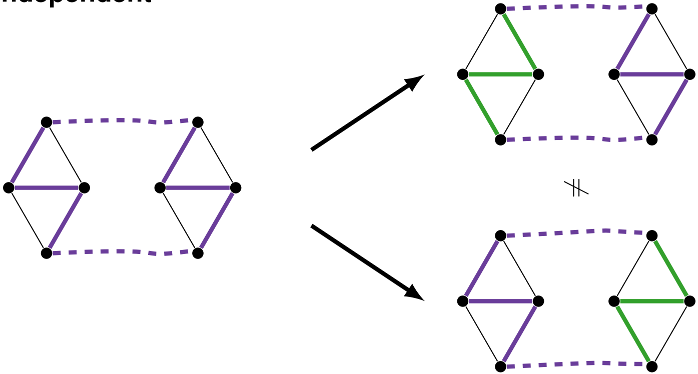


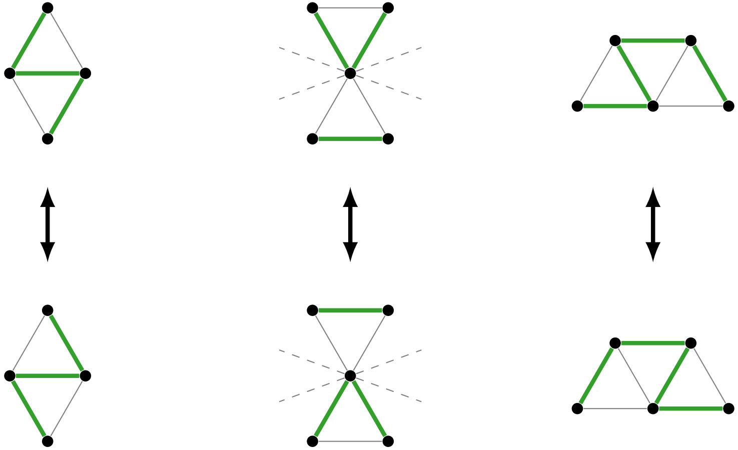
Counting bases

Very informally:

The counting base lemma (weak variant)

If one has a counting base with a set S of switching subgraphs so that each switching subgraph overlaps with at most c others, then there are at least $\frac{|S|}{c}$ hamiltonian cycles.

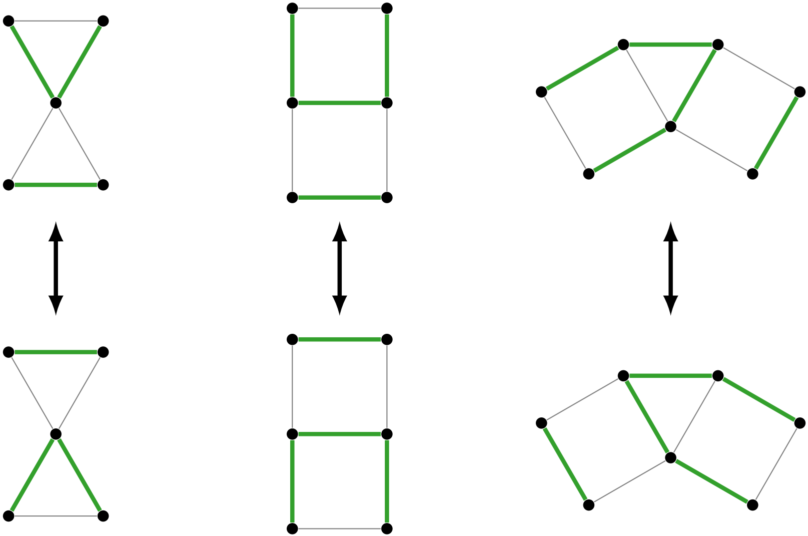
Switching subgraphs for triangulations



Counting base for 4-connected polyhedra

Problem: polyhedra can locally look very different.

Switching subgraphs for 4-connected polyhedra



Counting base for 4-connected polyhedra

The conditions **closed** and **independent** are easily verified, so only **saturation** needs to be examined.

The tool to solve this is:

Lemma (Jackson, Yu, 2002)

Let (G, F) be a circuit graph, r, z be vertices of G and $e \in E(F)$. Then G contains an F-Tutte cycle X through e, r and z.

Circuit graph: G plane, 2-connected, F facial cycle, for each 2-cut each component contains elements from F

F-Tutte cycle: cycle C, so that bridges contain at most 3 endpoints on C and at most 2 if it contains an edge of F.

Counting base for 4-connected polyhedra

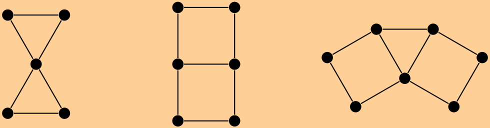
Unfortunately. . .

- for each such switching subgraph there are 4-connected polyhedra not containing it
- for each pair of those switching subgraphs there are 4-connected polyhedra containing only a small constant number of them

but. . .

Theorem

Each 4-connected polyhedron has a linear number of the three switching subgraphs below.



So, applying the counting base lemma:

Theorem

4-connected polyhedra have at least a linear number of hamiltonian cycles.

Linear number of switching subgraphs

Let f_i denote the number of faces of size i.

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4) f_i$$

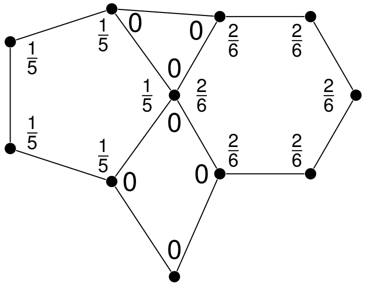
Linear number of switching subgraphs

Linear number of switching subgraphs

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4) f_i$$

- Assign the value 0 to angles of triangles and quadrangles
- Assign the value $\frac{i-4}{i}$ to each angle of an i-gon with $i > 4$

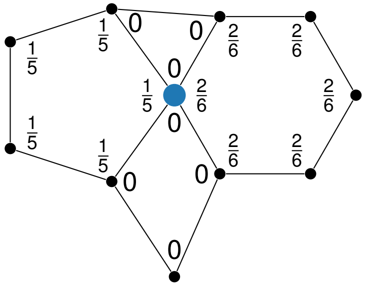


Linear number of switching subgraphs

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4) f_i$$

Define $a(v)$ as the sum of all angle values around v.



Linear number of switching subgraphs

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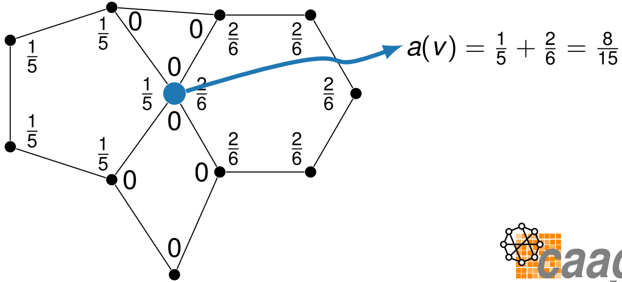


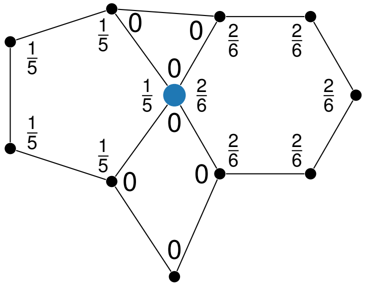
Linear number of switching subgraphs

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4) f_i$$

Define $a(v)$ as the sum of all angle values around v.

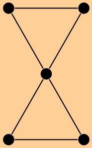
$$\Rightarrow \sum_{v \in V} a(v) = \sum_{i>4} (i-4) f_i$$



Linear number of switching subgraphs

Lemma

A polyhedron has at least $3f_3 - |V|$ *hourglasses.*

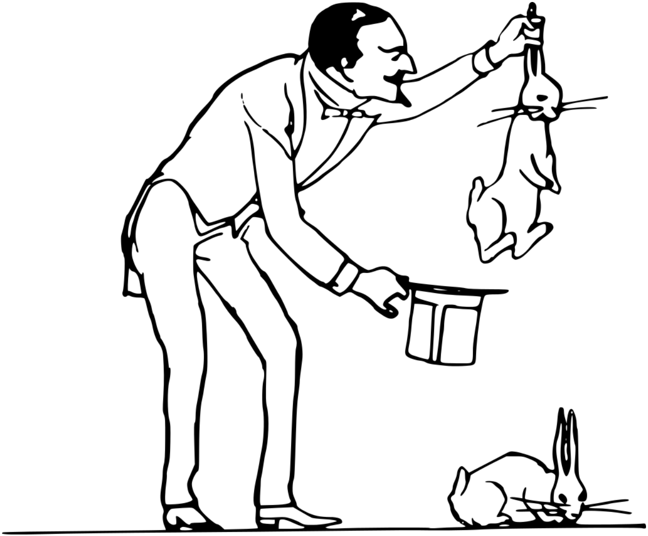


Let $\mathcal{S}$ denote the set of switching subgraphs.
Let $\mathcal{S}_H$ denote the set of hourglasses.

Lemma

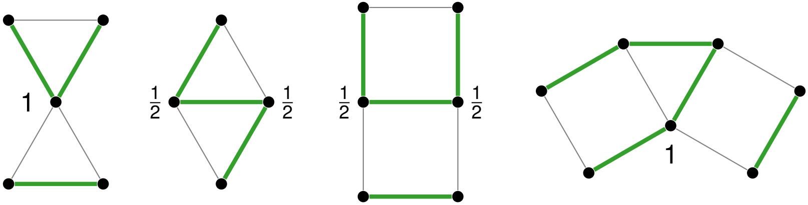
$$|\mathcal{S}| \geq |\mathcal{S}_H| \geq 24 + 3\sum_{v\in V} a(v) - |V|$$

Linear number of switching subgraphs



Linear number of switching subgraphs

Count the switching subgraphs in a special way:



Define $w(v)$ as the sum of all values at the vertex v.

$$\sum_{v \in V} w(v) = |\mathcal{S}|$$

Linear number of switching subgraphs

There are 4-connected polyhedra for which:

- the minimum of a over all vertices is 0, and…
- the minimum of w over all vertices is 0.

Linear number of switching subgraphs

Lemma

Let $G = (V, E)$ be a plane graph with minimum degree ≥ 4. Then for each $v \in V$ we have

$$a(v) + w(v) \geq \frac{2}{5}$$

so

$$\sum_{v \in V} a(v) + |\mathcal{S}| \geq \frac{2}{5}|V|$$

Linear number of switching subgraphs

Lemma

For 4-connected polyhedra we have

$$|\mathcal{S}| \geq \frac{1}{20}|V| + 6$$

Proof: Set $A(V) = \sum_{v \in V} a(v)$.

We have two lower bounds for $|S|$:

$$|S| \geq 24 + 3A(V) - |V|$$

$$|S| \geq \frac{2}{5}|V| - A(V)$$

Linear number of switching subgraphs

Lemma

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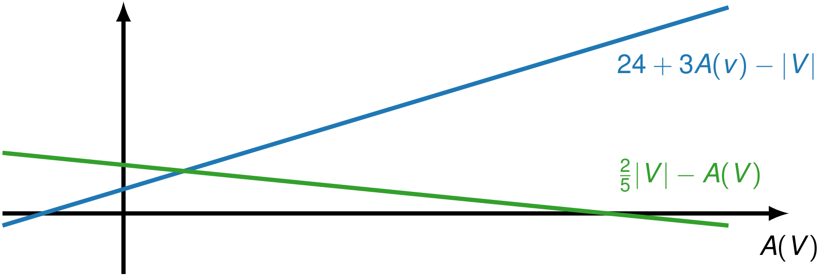
$$|\mathcal{S}| \geq \frac{1}{20}|V| + 6$$



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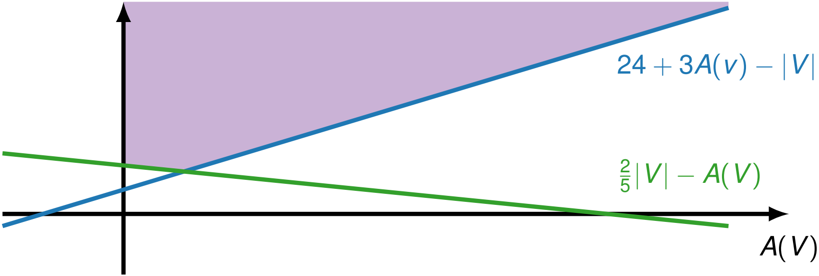

Linear number of switching subgraphs

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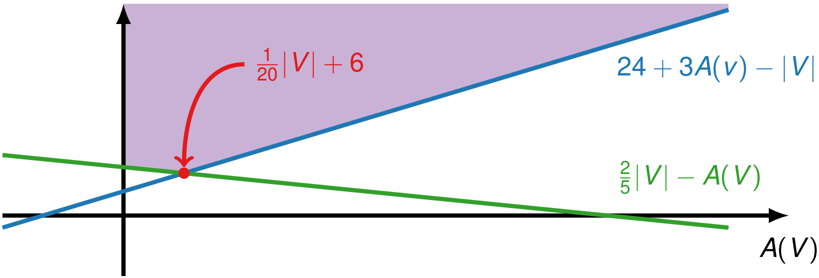

Theorem

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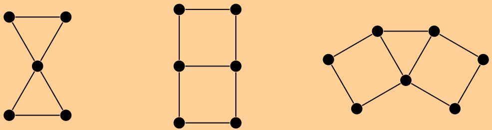


So, applying the counting base lemma:

Theorem

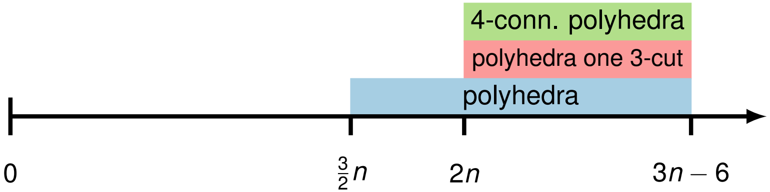
4-connected polyhedra have at least a linear number of hamiltonian cycles.

Few 3-cuts

Theorem

Let $c > 0$. Polyhedra with at most one 3-cut and at least $(2 + \frac{2}{33} + c)|V|$ edges have at least a linear number of hamiltonian cycles.

Edges in polyhedra on n vertices



Summary

Theorem

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