

4-connected polyhedra have at least a linear number of hamiltonian cycles

Gunnar Brinkmann Nico Van Cleemput

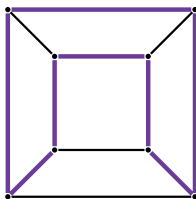
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 - Counting base
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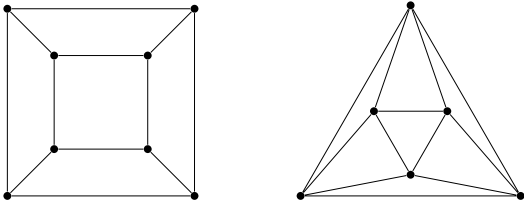
Hamiltonian cycles



A hamiltonian cycle is a spanning cycle.



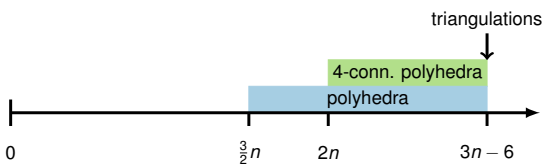
Polyhedra and triangulations



- Polyhedra are 3-connected plane graphs
- A triangulation is a polyhedron with only triangular faces



Edges in polyhedra on n vertices



More edges suggests: more likely to be hamiltonian!



90 years of theorems

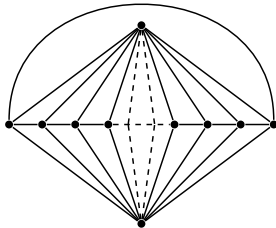
Triangulations		Polyhedra
4-conn. \Rightarrow hamiltonian Whitney (1931)	$\leftarrow 25 \text{ years} \rightarrow$	4-conn. \Rightarrow hamiltonian Tutte (1956)
at most three 3-cuts \Rightarrow hamiltonian Jackson, Yu (2002)	$\leftarrow 17 \text{ years} \rightarrow$	at most three 3-cuts \Rightarrow hamiltonian Brinkmann, Zamfirescu (2019)
six 3-cuts can be non-hamiltonian		six 3-cuts can be non-hamiltonian
four or five 3-cuts: unknown, but 1-tough		four or five 3-cuts: unknown, but 1-tough

Number of hamiltonian cycles

4-connected triangulations	4-connected polyhedra
≥ 1 hamiltonian cycle Whitney (1931)	≥ 1 hamiltonian cycle Tutte (1956)
$\geq \frac{n}{\log n}$ hamiltonian cycles Hakimi, Schmeichel, Thomassen (1979)	
	≥ 6 hamiltonian cycles Thomassen (1983)
$\geq \frac{12}{5}(n - 2)$ hamiltonian cycles Brinkmann, Souffriau, VC (2018)	
$\geq \frac{161}{60}(n - 2)$ hamiltonian cycles Brinkmann, Cuvelier, VC (2018)	

Number of hamiltonian cycles

- Up to 17 vertices there are 4-connected polyhedra with fewer hamiltonian cycles than the double wheel
- For 18 vertices or more the double wheel appears to be the polyhedron with the fewest number of hamiltonian cycles



$2(n - 2)(n - 4)$ hamiltonian cycles

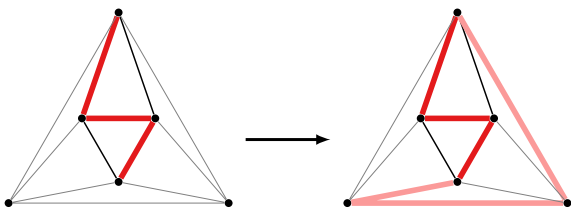


Hakimi, Schmeichel, Thomassen (1979)

Using a result of Whitney (1931):

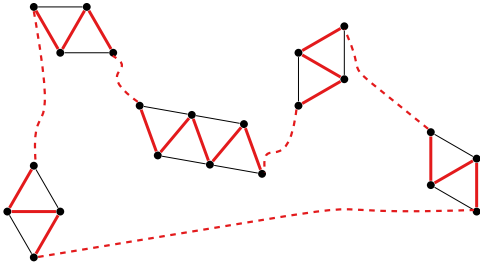
Lemma

Each zigzag in a 4-connected triangulation can be extended to a hamiltonian cycle.



Hakimi, Schmeichel, Thomassen (1979)

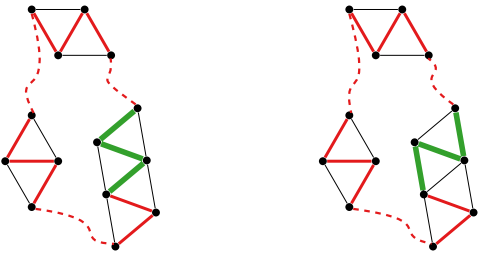
There is a linear number of such zigzags, but . . .



. . . a single hamiltonian cycle can contain a linear number of these zigzags, giving in total a constant number of hamiltonian cycles.

Hakimi, Schmeichel, Thomassen (1979)

A hamiltonian cycle with k disjoint zigzags guarantees 2^k hamiltonian cycles by 'switching'.



This explains the $\frac{1}{\log n}$ in the formula.

Counting bases

The main contribution of the 2018-paper:

counting differently via counting bases



Counting bases

Definition

Let G be a graph and let \mathcal{C} be a collection of hamiltonian cycles of G . The pair (S, r) , where $S \subseteq 2^{E(G)}$ and r is a function $r : S \rightarrow 2^{E(G)}$, is called a *counting base* if the pair (S, r) has the following properties:

- (i) for all $S \in S$, there is a hamiltonian cycle $C \in \mathcal{C}$ saturating S .
- (ii) for all $S \in S$, $r(S) \subseteq E(G)$ (not necessarily in S) so that $S \not\subseteq r(S)$ and for each hamiltonian cycle $C \in \mathcal{C}$, if $S \cap C \neq \emptyset$ we have that $z(C, S) = (C \setminus S) \cup r(S)$ is a hamiltonian cycle.
- (iii) for all $S_1 \neq S_2$, $S_1, S_2 \in S$ and C saturating S_1 and S_2 , we have that $z(C, S_1) \neq z(C, S_2)$.

Don't read this slide!



Counting bases

A *counting base* is a set of subgraphs (*switching subgraphs*) together with a function (*switching function*) satisfying 3 conditions:

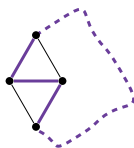
- (i) saturated
- (ii) closed
- (iii) independent



Counting bases

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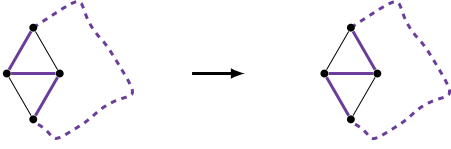
- (i) **saturated**
- (ii) closed
- (iii) independent



Counting bases

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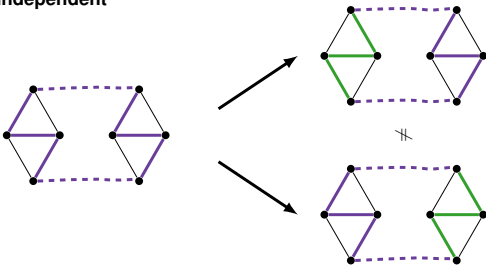
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Counting bases

A *counting base* is a set of subgraphs (*switching subgraphs*) together with a function (*switching function*) satisfying 3 conditions:

- (i) saturated
- (ii) closed
- (iii) independent



Counting bases

Very informally:

The counting base lemma (weak variant)

If one has a counting base with a set S of switching subgraphs so that each switching subgraph overlaps with at most c others, then there are at least $\frac{|S|}{c}$ hamiltonian cycles.



Introduction 4-connected polyhedra Few 3-cuts Summary Definitions Hamiltonian cycles Counting base

Switching subgraphs for triangulations

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Introduction 4-connected polyhedra Few 3-cuts Summary Counting base Linear number

Counting base for 4-connected polyhedra

Problem: polyhedra can locally look very different.

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Switching subgraphs for 4-connected polyhedra

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Counting base for 4-connected polyhedra

The conditions **closed** and **independent** are easily verified, so only **saturation** needs to be examined.

The tool to solve this is:

Lemma (Jackson, Yu, 2002)

Let (G, F) be a circuit graph, r, z be vertices of G and $e \in E(F)$. Then G contains an F -Tutte cycle X through e, r and z .

Circuit graph: G plane, 2-connected, F facial cycle, for each 2-cut each component contains elements from F

F-Tutte cycle: cycle C , so that **bridges** contain at most 3 endpoints on C and at most 2 if it contains an edge of F .



Counting base for 4-connected polyhedra

Unfortunately...

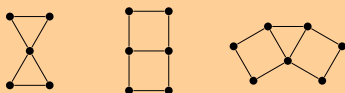
- for each such switching subgraph there are 4-connected polyhedra not containing it
- for each pair of those switching subgraphs there are 4-connected polyhedra containing only a small constant number of them

but...



Theorem

Each 4-connected polyhedron has a linear number of the three switching subgraphs below.



So, applying the counting base lemma:

Theorem

4-connected polyhedra have at least a linear number of hamiltonian cycles.

Linear number of switching subgraphs

Let f_i denote the number of faces of size i .

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4)f_i$$



Linear number of switching subgraphs

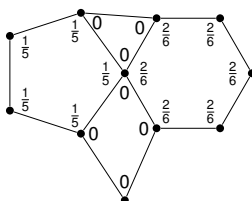


Linear number of switching subgraphs

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4)f_i$$

- Assign the value 0 to angles of triangles and quadrangles
- Assign the value $\frac{i-4}{i}$ to each angle of an i -gon with $i > 4$

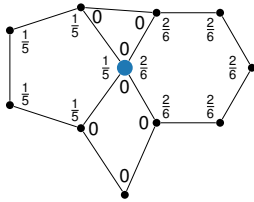


Linear number of switching subgraphs

Lemma

$$f_3 \geq 8 + \sum_{i>4} (i-4)f_i$$

Define $a(v)$ as the sum of all angle values around v .

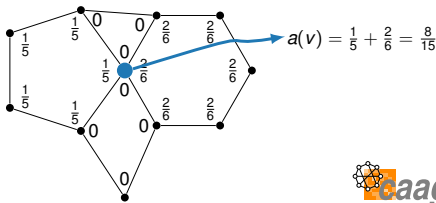


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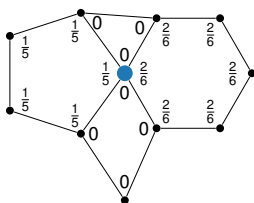
Linear number of switching subgraphs

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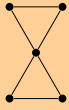
$$\Rightarrow \sum_{v \in V} a(v) = \sum_{i>4} (i-4)f_i$$



Linear number of switching subgraphs

Lemma

A polyhedron has at least $3f_3 - |V|$ hourglasses.

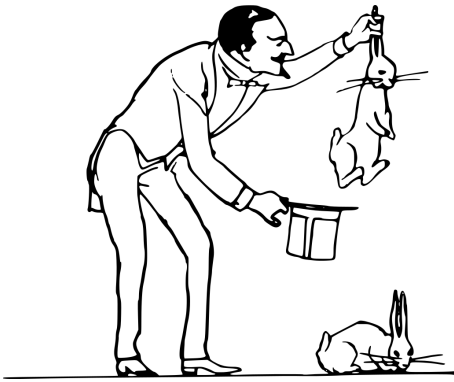


Let S denote the set of switching subgraphs.
 Let S_H denote the set of hourglasses.

Lemma

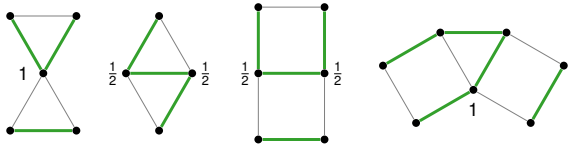
$$|S| \geq |S_H| \geq 24 + 3 \sum_{v \in V} a(v) - |V|$$

Linear number of switching subgraphs



Linear number of switching subgraphs

Count the switching subgraphs in a special way:



Define $w(v)$ as the sum of all values at the vertex v .

$$\sum_{v \in V} w(v) = |S|$$

Linear number of switching subgraphs

There are 4-connected polyhedra for which:

- the minimum of a over all vertices is 0, and...
- the minimum of w over all vertices is 0.



Linear number of switching subgraphs

Lemma

Let $G = (V, E)$ be a plane graph with minimum degree ≥ 4 .
Then for each $v \in V$ we have

$$a(v) + w(v) \geq \frac{2}{5}$$

so

$$\sum_{v \in V} a(v) + |S| \geq \frac{2}{5} |V|$$



Linear number of switching subgraphs

Lemma

For 4-connected polyhedra we have

$$|S| \geq \frac{1}{20} |V| + 6$$

Proof: Set $A(V) = \sum_{v \in V} a(v)$.

We have two lower bounds for $|S|$:

$$|S| \geq 24 + 3A(V) - |V|$$

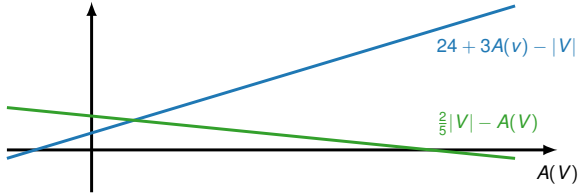
$$|S| \geq \frac{2}{5} |V| - A(V)$$

Linear number of switching subgraphs

Lemma

For 4-connected polyhedra we have

$$|S| \geq \frac{1}{20}|V| + 6$$

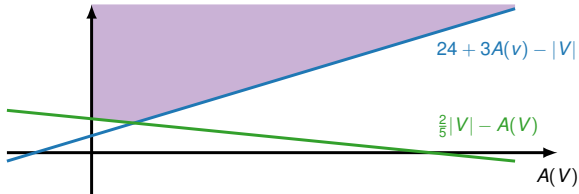


Linear number of switching subgraphs

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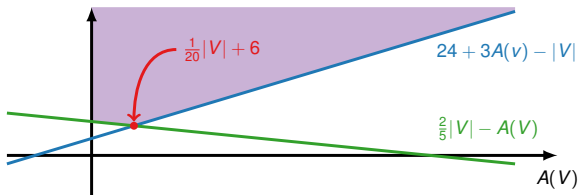


Linear number of switching subgraphs

Lemma

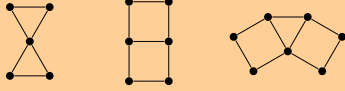
For 4-connected polyhedra we have

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Theorem

Each 4-connected polyhedron has a linear number of the three switching subgraphs below.



So, applying the counting base lemma:

Theorem

4-connected polyhedra have at least a linear number of hamiltonian cycles.

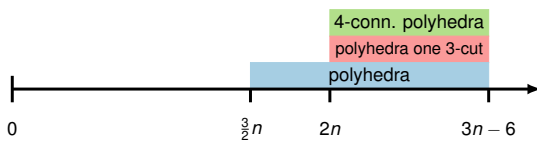
Few 3-cuts

Theorem

Let $c > 0$. Polyhedra with at most one 3-cut and at least $(2 + \frac{c}{33} + c)|V|$ edges have at least a linear number of hamiltonian cycles.



Edges in polyhedra on n vertices



Summary

Theorem

4-connected polyhedra have at least a linear number of hamiltonian cycles.

Theorem

Let $c > 0$. Polyhedra with at most one 3-cut and at least $(2 + \frac{2}{33} + c)|V|$ edges have at least a linear number of hamiltonian cycles.