Shortness coefficient of cyclically 4-edge-connected cubic graphs

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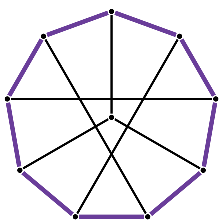
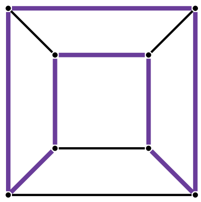
Circumference



The circumference $circ(G)$ is the length of a longest cycle.

Hamiltonicity



A graph G is hamiltonian if $circ(G) = |V(G)|$.

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Hamiltonicity of classes of graphs

- Tait conjectured in 1884 that every cubic polyhedron is hamiltonian.
- The conjecture became famous because it implied the Four Colour Theorem (at that time still the Four Colour Problem)

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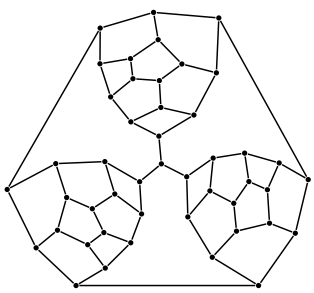
Hamiltonicity of classes of graphs



The first to construct a counterexample was Tutte in 1946

Hamiltonicity of classes of graphs

Theorem (Tutte, 1956)

Every 4-connected polyhedron is hamiltonian.

Hamiltonicity of classes of graphs

How far is a class of graphs from being hamiltonian?

Shortness coefficient

The **shortness coefficient** of $\mathcal{G}$ is defined as

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\mathrm{circ}(G)}{|V(G)|}$$

with lim inf taken over all sequences of graphs G_n in $\mathcal{G}$ such that $|V(G_n)| \to \infty$ for $n \to \infty$.

Shortness coefficient

$$\rho(\mathcal{G}) = \liminf_{G \in \mathcal{G}} \frac{\operatorname{circ}(G)}{|V(G)|}$$

- $0 \leq \rho(\mathcal{G}) \leq 1$
- every graph in $\mathcal{G}$ is hamiltonian $\Rightarrow \rho(\mathcal{G}) = 1$

Known results

Theorem (Moon and Moser, 1963)

The shortness coefficient of the class of 3-connected planar graphs is 0.

Theorem (Tutte, 1956)

The shortness coefficient of the class of 4-connected planar graphs is 1.

Known results

Theorem (Bondy and Simonovits, 1980)

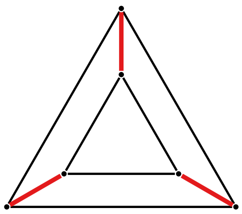
The shortness coefficient of the class of 3-connected cubic graphs is 0.

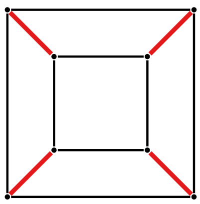
Theorem (Walther, 1969)

The shortness coefficient of the class of 3-connected cubic planar graphs is 0.

Cyclically k-edge-connected

A graph G is cyclically k-edge-connected if for every edge-cut S of G with less than k edges at most one component of $G - S$ contains a cycle.

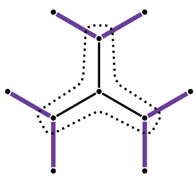




Cyclically k-edge-connected

- For $k \in \{1, 2, 3\}$ being cyclically k-edge-connected and being k-connected are equivalent for cubic graphs.
- $\mathcal{C}k$ is the class of cyclically k-edge-connected cubic graphs.
- $\mathcal{C}k\mathcal{P}$ is the class of cyclically k-edge-connected planar cubic graphs.

Known bounds



$$circ(G) \geq \frac{3}{4}|V(G)|$$

Known bounds

Theorem (Grünbaum and Malkevitch, 1976)

$\rho(C4\mathcal{P}) \leq \frac{76}{77}$

Theorem (Lo and Schmidt, 2018)

$\rho(C4\mathcal{P}) \leq \frac{52}{53}$

Question

$\rho(C4\mathcal{P}) \leq \frac{41}{42}$?

A new bound

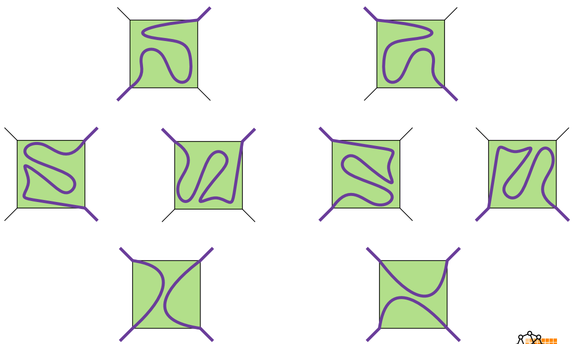
Theorem (Lo, Schmidt, VC, and Zamfirescu)

$\rho(C4\mathcal{P}) \leq \frac{37}{38}$

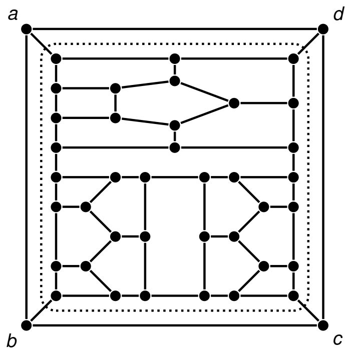
Approach

- Find cyclically 4-edge-connected fragments such that (almost) any intersection with a cycle misses some vertices.
- Combine these fragments to construct an infinite family of graphs obtaining the bound in the limit.

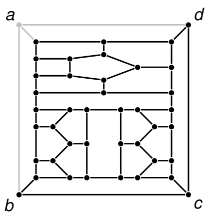
Fragments and cycles

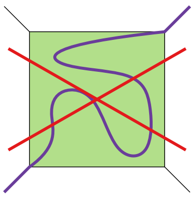


A new bound



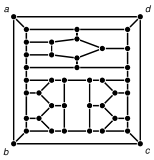
A new bound





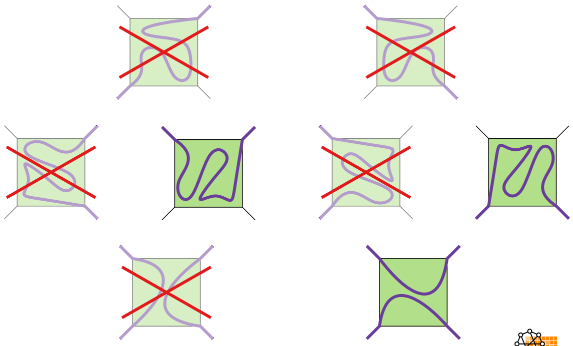
$H - a$ is non-hamiltonian

A new bound

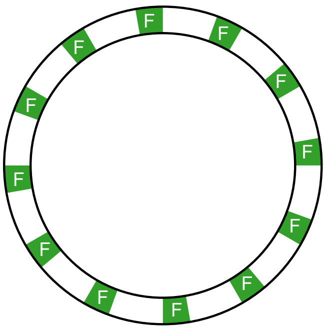


- $H - a$ is non-hamiltonian
- $H - d$ is non-hamiltonian
- $H - a - b$ is non-hamiltonian
- $H - c - d$ is non-hamiltonian
- $H - ab - cd$ is non-hamiltonian

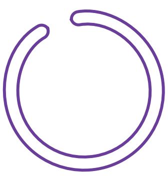
A new bound



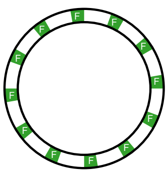
A new bound



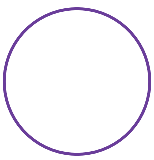
A new bound



misses at least
$k-2$ vertices



k copies of fragment



misses at least
k vertices

A new bound

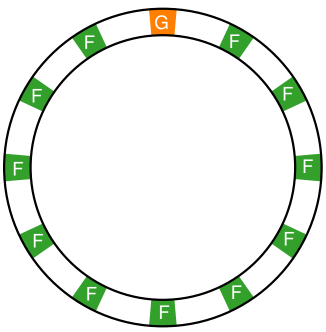
$$\rho(\mathcal{C}4\mathcal{P}) = \liminf_{G \in \mathcal{C}4\mathcal{P}} \frac{\operatorname{circ}(G)}{|V(G)|} \leq \lim_{k \to \infty} \frac{38k - (k-2)}{38k} = \frac{37}{38}$$

Higher genus

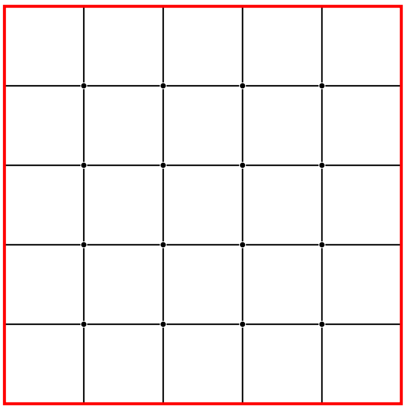
Theorem (Lo, Schmidt, VC, and Zamfirescu)

For every $g \geq 0$, the shortness coefficient of the class of cyclically 4-edge-connected cubic graphs of genus g is at most $\frac{37}{38}$.

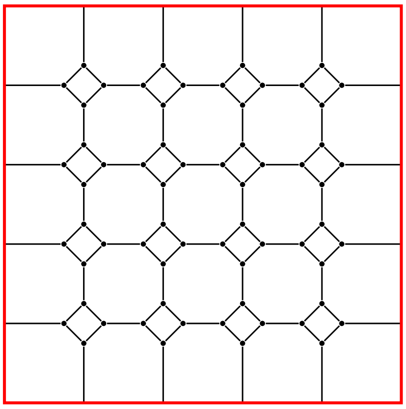
Increasing the genus



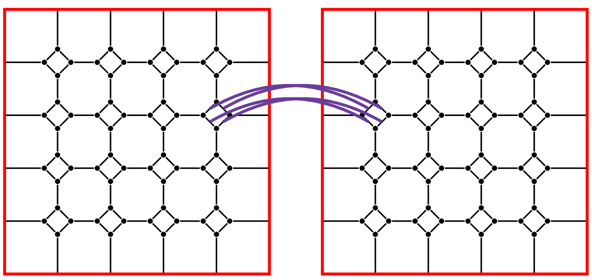
A fragment with arbitrary genus



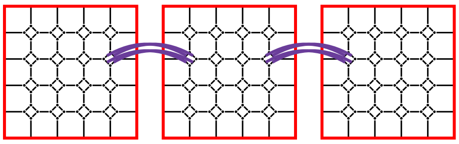
A fragment with arbitrary genus



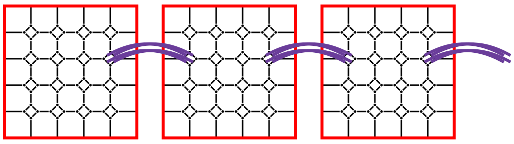
A fragment with arbitrary genus



A fragment with arbitrary genus



A fragment with arbitrary genus

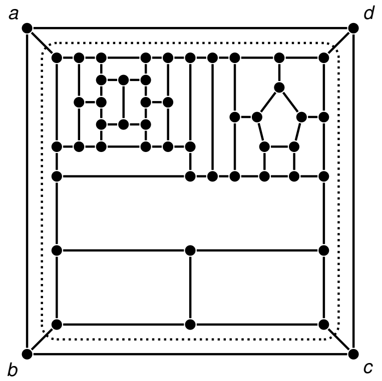


Bounded face length

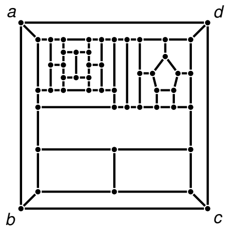
Theorem (Lo, Schmidt, VC, and Zamfirescu)

For all $\ell \geq 23$, the shortness coefficient of the class of cyclically 4-edge-connected cubic plane graphs with faces of length at most ℓ is at most $\frac{45}{46}$.

A second fragment

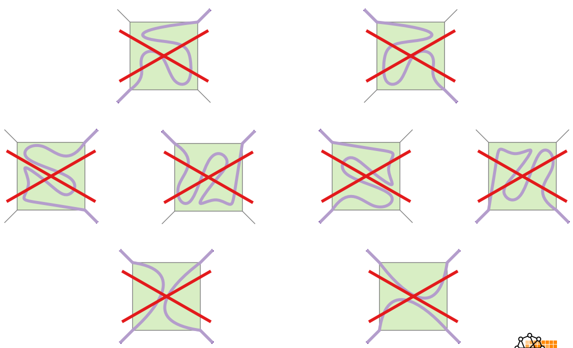


A second fragment



- H is not hamiltonian
- $H - a$ is not hamiltonian
- $H - d$ is not hamiltonian

A second fragment

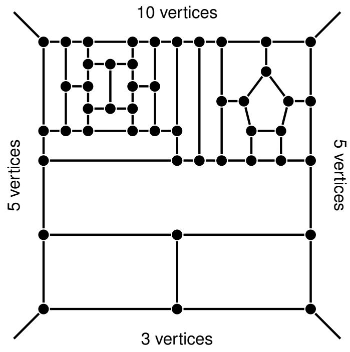


A new bound

Replacing each vertex of a 4-connected 4-regular planar graph on k vertices by this fragment results in a cyclically 4-edge-connected cubic planar graph in which each cycle spanning multiple fragments misses at least one vertex in each fragment.

$$\rho(\mathcal{C}4\mathcal{P}) = \liminf_{G \in \mathcal{C}4\mathcal{P}} \frac{\operatorname{circ}(G)}{|V(G)|} \leq \lim_{k \to \infty} \frac{45k}{46k} = \frac{45}{46}$$

Bounded face length



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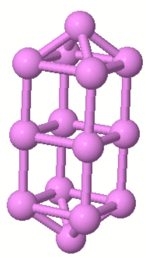

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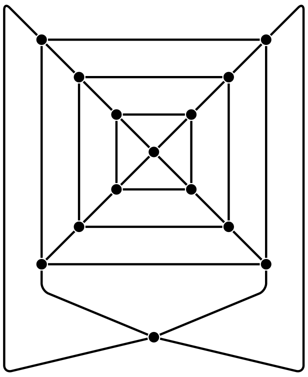
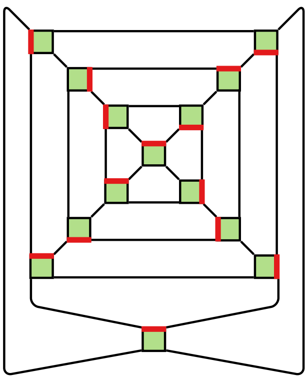
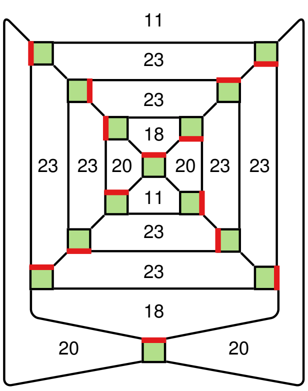


Bounded face length



Bounded face length



Increasing the genus

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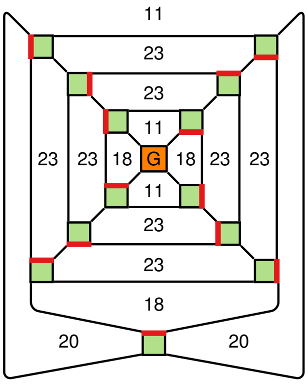
Increasing the genus



General cubic graphs

Theorem (Lo, Schmidt, VC, and Zamfirescu)

Let G be a cyclically 4-edge-connected cubic graph on n vertices.

Then $\rho(\mathcal{C}4) \leq \dfrac{circ(G) - 2}{n - 2}$, and if there exist adjacent vertices v, w in G such that $G - v - w$ is planar, then $\rho(\mathcal{C}4\mathcal{P}) \leq \dfrac{circ(G) - 2}{n - 2}$.

Corollary

$\rho(\mathcal{C}4) \leq \frac{7}{8}$ and $\rho(\mathcal{C}4\mathcal{P}) \leq \frac{39}{40}$.

Ongoing/future work

- $\frac{3}{4} \leq \rho(\mathcal{C}4\mathcal{P}) \leq \frac{37}{38}$
  - shrink the gap
  - fragments are smallest possible
  - missing more vertices
- shortness exponent of quartic/quintic polyhedra and polyhedra with two types of degrees