Generation of Delaney-Dress symbols

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Delaney-Dress symbols?

A Delaney-Dress symbol encodes an equivariant tiling
(i.e. a tiling together with its symmetry group)

The main characters

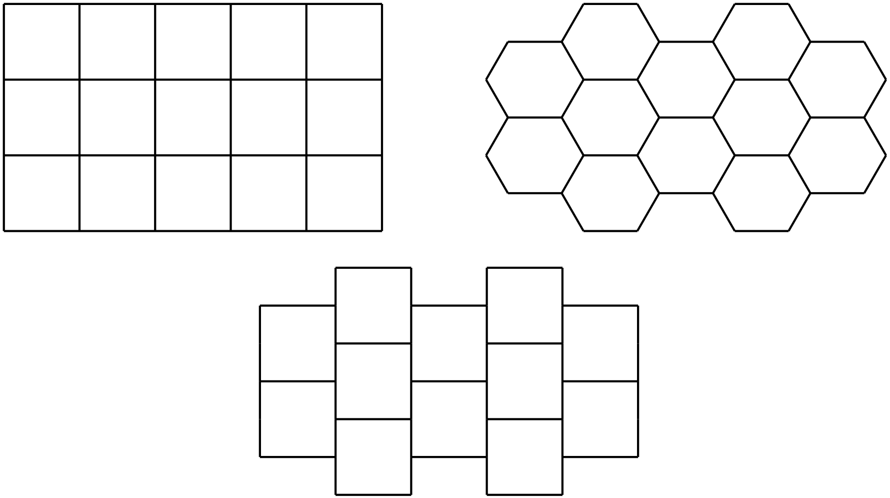
Definition

Tiling T = set of tiles $t_1, t_2, \ldots$ with $t_i \subset \mathbb{E}^2$, t_i homeomorph to $\overline{B}(0,1)$ that satisfy the following conditions:

1. $\bigcup_{t \in T} t = \mathbb{E}^2$
2. $\forall t_i, t_j (i \neq j) \in T : t_i^\circ \cap t_j^\circ = \emptyset \wedge t_i \cap t_j \in \{\emptyset, \{\text{points}\}, \{\text{lines}\}\}.$
3. $\forall x \in \mathbb{E}^2 : x$ has a neighbourhood that only intersects a finite number of tiles.

Periodic tiling

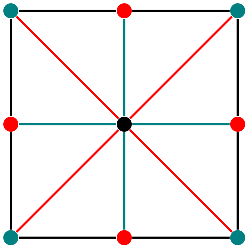
symmetry group contains *two independent translations*



Barycentric subdivision

- For each face: one point
- For each edge: one point
- For each vertex: one point

Incidence determines adjacency

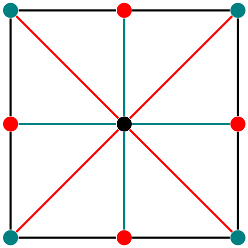


Chamber system

Define $\Sigma = \langle \sigma_0, \sigma_1, \sigma_2 | \sigma_i^2 = \mathbb{1} \rangle$

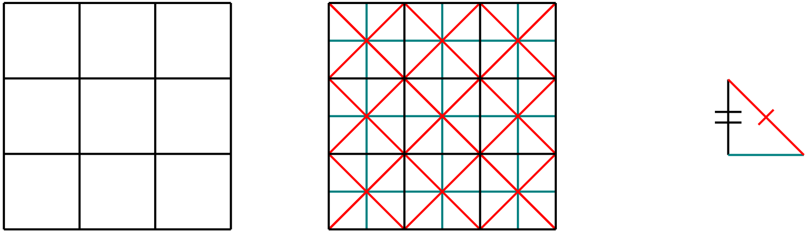
σ_0 : change the green point (vertex).
σ_1 : change the red point (edge).
σ_2 : change the black point (face).

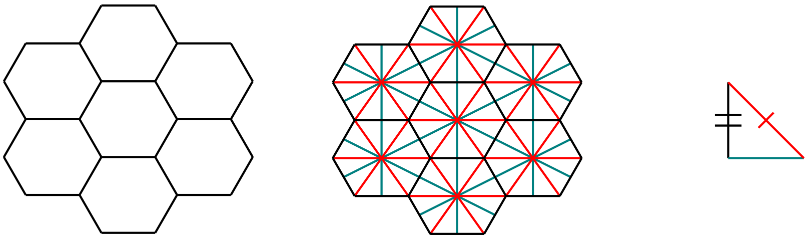
Chamber system $\mathcal{C}$ of T = barycentric subdivision with Σ



Delaney-Dress graph

The Delaney-Dress graph $\mathcal{D}$ of a periodic tiling is the set of *orbits of the chambers* of the chamber system of the tiling under the symmetry group of the tiling.





Observation

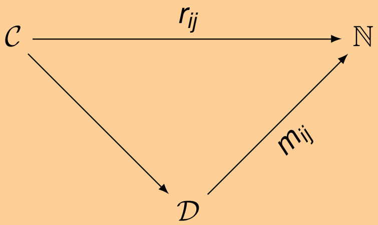
Delaney-Dress graph is not sufficient to distinguish between tilings!

Define functions $r_{ij} : \mathcal{C} \to \mathbb{N}; c \mapsto r_{ij}(c)$ with $r_{ij}(c)$ the smallest number for which $c(\sigma_i \sigma_j)^{r_{ij}(c)} = c$.

Observation

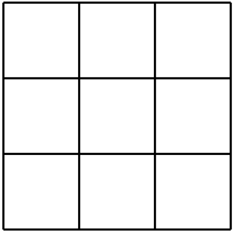
r_{02} is a constant function with value 2.
$r_{01}(c)$ is the size of the face of T that belongs to c.
$r_{12}(c)$ is the number of faces that meet in the vertex that belongs to c.

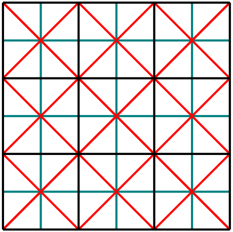
Define functions $m_{ij} : \mathcal{D} \to \mathbb{N}; d \mapsto m_{ij}(c)$ in such a manner that the following diagram is commutative:

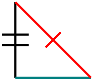


Delaney-Dress symbol

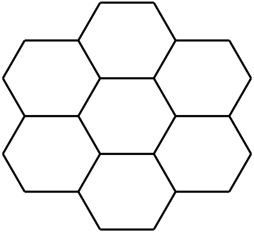
The Delaney-Dress symbol of a periodic tiling is $(\mathcal{D}; m_{01}, m_{12})$.

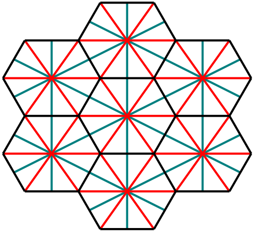


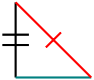




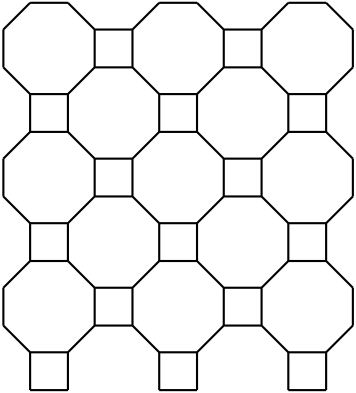
$$m_{01}(c) = 4$$
$$m_{12}(c) = 4$$

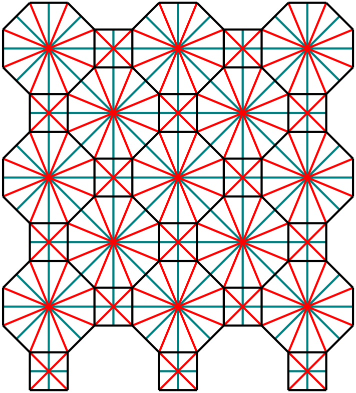


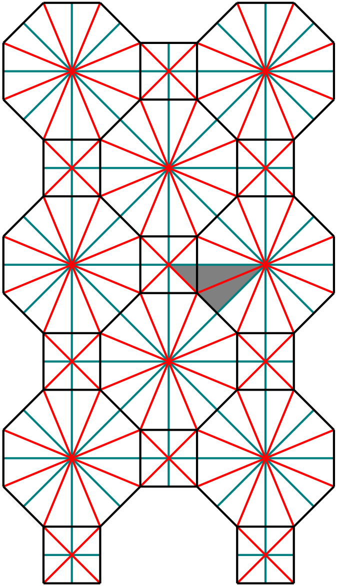


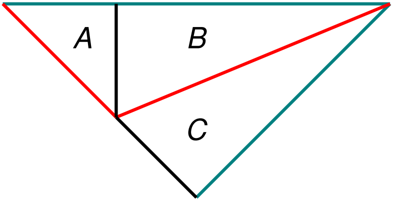


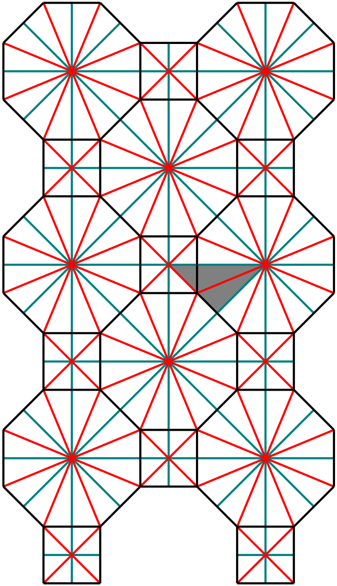
$$m_{01}(c) = 6$$
$$m_{12}(c) = 3$$

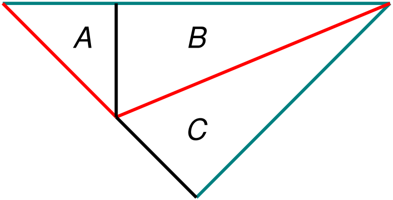




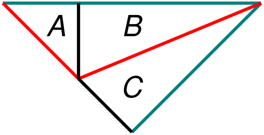


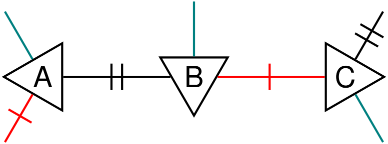

A
B
C





	m_{01}	m_{12}
A	4	3
B	8	3
C	8	3





	m_{01}	m_{12}
A	4	3
B	8	3
C	8	3

Theorem

$(\mathcal{D}; m_{01}, m_{12})$ is the Delaney-Dress symbol of a periodic tiling iff

1. *$\mathcal{D}$ is finite*
2. *Σ works transitively on $\mathcal{D}$*
3. *m_{01} is constant on $\langle \sigma_0, \sigma_1 \rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_0\sigma_1)^{m_{01}(d)} = d$*
4. *m_{12} is constant on $\langle \sigma_1, \sigma_2 \rangle$ orbits and $\forall d \in \mathcal{D} : d(\sigma_1\sigma_2)^{m_{12}(d)} = d$*
5. *$\forall d \in \mathcal{D} : d(\sigma_0\sigma_2)^2 = d$*
6. *Curvature condition*

caagt

Curvature condition

$$K(\mathcal{D}) = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right)$$

- $K(\mathcal{D}) < 0 \rightarrow$ hyperbolic plane
- $K(\mathcal{D}) = 0 \rightarrow$ euclidean plane
- $K(\mathcal{D}) > 0 \rightarrow$ sphere iff $\frac{4}{K(\mathcal{D})} \in \mathbb{N}$

Motivation

- Can we generate all cubic pregraphs?
  (i.e. multigraphs with loops and semi-edges)
- Can we filter out the 3-edge-colourable ones?
- Can we filter out the Delaney-Dress graphs?

Why this wasn't the best modus operandi

n	colourable	Delaney-Dress	ratio
1	1	1	100.00 %
2	3	3	100.00 %
3	3	2	66.67%
4	11	9	81.82%
5	17	7	41.18%
6	59	29	49.15%
7	134	27	20.15%
8	462	105	22.73%
9	1 332	118	8.86%
10	4 774	392	8.21%
11	16 029	546	3.41%
12	60 562	1 722	2.84%
13	225 117	2 701	1.20%
14	898 619	7 953	0.89%
15	3 598 323	13 966	0.39%
16	15 128 797	40 035	0.26%
17	64 261 497	75 341	0.12%
18	283 239 174	210 763	0.07%
19	1 264 577 606	420 422	0.03%
20	5 817 868 002	1 162 192	0.02%

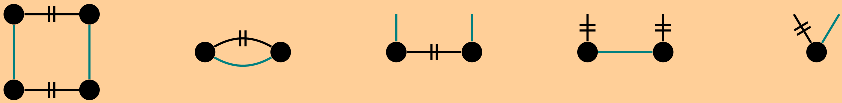
The structure of Delaney-Dress graphs

From the theorem...

1. $\mathcal{D}$ is finite
2. Σ works transitively on $\mathcal{D}$
5. $\forall d \in \mathcal{D} : d(\sigma_0\sigma_2)^2 = d$

Translated:

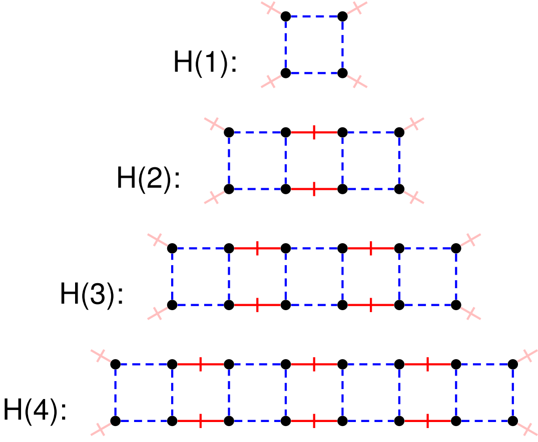
Finite, connected, 3-edge-coloured pregraphs where each 02-component is isomorph to one of

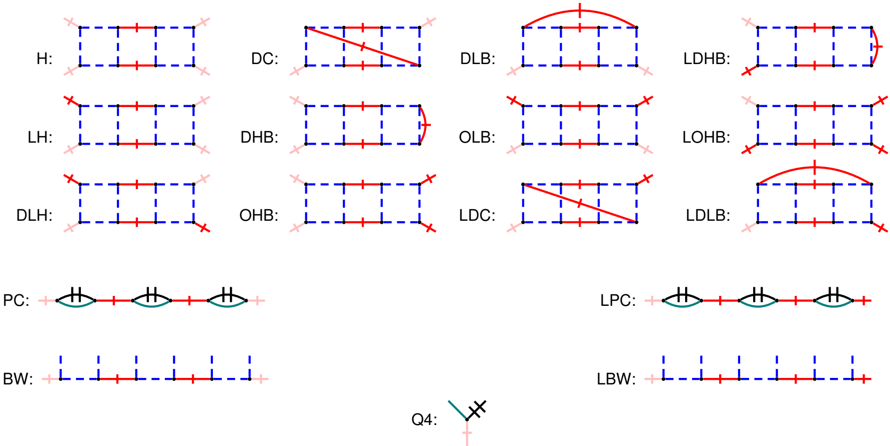


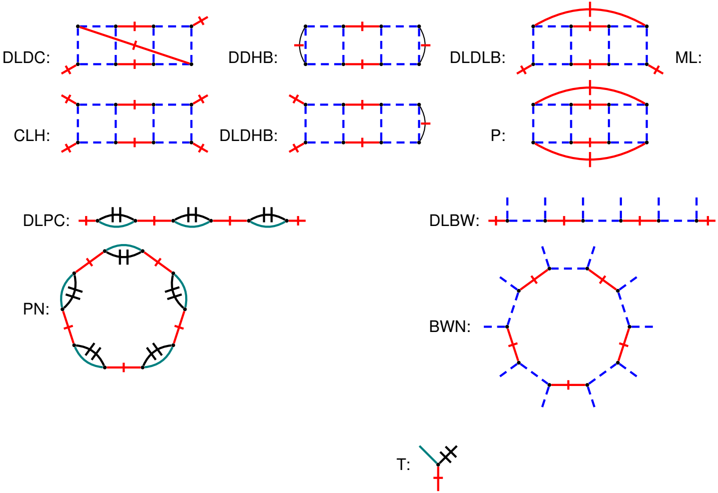
q_1 q_2 q_3 q_3 q_4

There exists for each Delaney-Dress graph a unique partition of the graph into subgraphs of some specific types

Example of such a parameterized type:



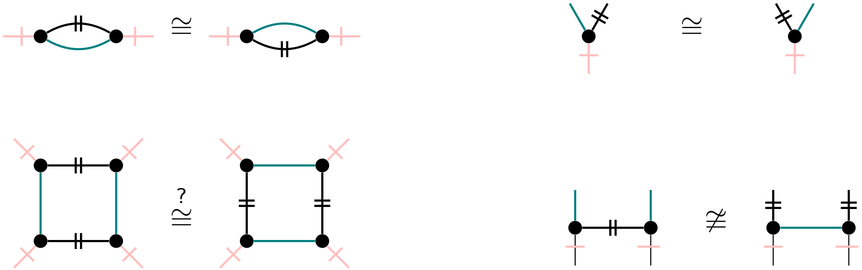

H:
DC:
DLB:
LDHB:
LH:
DHB:
OLB:
LOHB:
DLH:
OHB:
LDC:
LDLB:
PC:
LPC:
BW:
LBW:
Q4:


DLDC:
DDHB:
DLDLB:
ML:
CLH:
DLDHB:
P:
DLPC:
DLBW:
PN:
BWN:
T:

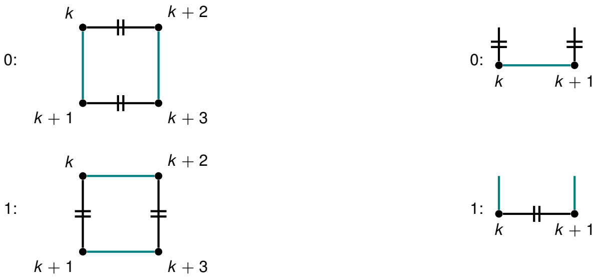
Enumerating the Delaney-Dress symbols

1. Generate lists of blocks
2. Connect blocks in list
3. Assign missing colours
4. Determine functions m_{01} and m_{12}

The last colours



Numbering in 02-components is consistent.



⇒ colour assignment represented by bit vector

Detecting isomorphic colour assignments

Question

Given an automorphism ϕ of the graph without coloured q_1 and q_3 components and a bit vector representing a colour assignment: what is the image of that assignment?

Tools

- vertices $V = \{1, ..., n\}$ with n the order of the graph
- set of q_1 and q_3 components Q
- bitvector B: B_q ($q \in Q$) is the position corresponding to the component q
- $v2q : V \to Q \cup \infty; v \mapsto \begin{cases} q & \text{if } v \in q \\ \infty & \text{in all other cases.} \end{cases}$
- $q2v : Q \to V \cup \infty; q \mapsto \min\{v \in V | v \in q\}$

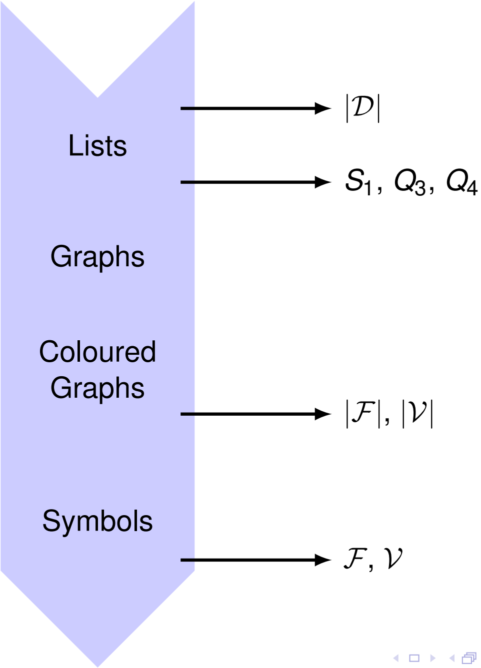
- q is q_3 component: $B'_{v2q(\phi(q2v(q)))} = B_q$
- q is q_1 component: define

$$w_1 := \phi(q2v(q))$$
$$w_2 := \phi(q2v(q) + 1)$$

  - if $|w_1 - w_2| = 1$: $B'_{v2q(w_1)} = B_q$
  - if $|w_1 - w_2| = 2$: $B'_{v2q(w_1)} = 1 - B_q$

A step back: limiting the graphs

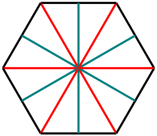
Property	Description
$\lvert\mathcal{D}\rvert$	The number of flags in the Delaney-Dress graph
$\mathcal{F}$	The multiset of sizes of a face in the face orbits
$\mathcal{V}$	The multiset of degrees of a vertices in the vertex orbits
$\lvert\mathcal{F}\rvert$	The number of face orbits
$\lvert\mathcal{V}\rvert$	The number of vertex orbits
S_1	The number of semi-edges with colour 1
Q_3	The number of q_3 components
Q_4	The number of q_4 components


Lists
Graphs
Coloured Graphs
Symbols
$|\mathcal{D}|$
S_1, Q_3, Q_4
$|\mathcal{F}|$, $|\mathcal{V}|$
$\mathcal{F}$, $\mathcal{V}$

Notation	Description	Default
$M_{\lvert\mathcal{F}\rvert}$	Maximum number of face orbits allowed	MAXN
$m_{\lvert\mathcal{F}\rvert}$	Minimum number of face orbits required	1
$M_{\lvert\mathcal{V}\rvert}$	Maximum number of vertex orbits allowed	MAXN
$m_{\lvert\mathcal{V}\rvert}$	Minimum number of vertex orbits required	1
$M_{\mathcal{F}}$	Maximum value of m_{01} allowed	$6 *$ MAXN
$m_{\mathcal{F}}$	Minimum value of m_{01} required	3
$M_{\mathcal{V}}$	Maximum value of m_{12} allowed	$6 *$ MAXN
$m_{\mathcal{V}}$	Minimum value of m_{12} required	3
$\mathcal{R}_{\mathcal{F}}$	Multiset of required face sizes	$\{\}$
$\mathcal{R}_{\mathcal{V}}$	Multiset of required vertex degrees	$\{\}$
$\mathcal{U}_{\mathcal{F}}$	Set of forbidden (unwanted) face sizes	$\{\}$
$\mathcal{U}_{\mathcal{V}}$	Set of forbidden (unwanted) vertex degrees	$\{\}$
$M_{\lvert\mathcal{D}\rvert}$	Maximum number of flags in the Delaney-Dress symbol	MAXN
$m_{\lvert\mathcal{D}\rvert}$	Minimum number of flags in the Delaney-Dress symbol	1

During the generation of the lists:

- limit the length of chains of digons
- in case of orientable tilings: exclude blocks containing semi-edges and odd cycles



$$K = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$0 \leq 2|\mathcal{F}| + \frac{|\mathcal{D}|}{m_{\mathcal{V}}} - \frac{|\mathcal{D}|}{2}$$

$$0 \leq 2M_{|\mathcal{F}|} + \frac{|\mathcal{D}|}{m_{\mathcal{V}}} - \frac{|\mathcal{D}|}{2}$$

$$|\mathcal{D}| \leq \left(\frac{4m_{\mathcal{V}}}{m_{\mathcal{V}} - 2} \right) M_{|\mathcal{F}|}$$

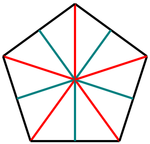
Similar:

$$|\mathcal{D}| \leq \left(\frac{4m_{\mathcal{F}}}{m_{\mathcal{F}} - 2} \right) M_{|\mathcal{V}|}$$

$$K = \sum_{d \in \mathcal{D}} \left(\frac{1}{m_{01}(d)} + \frac{1}{m_{12}(d)} - \frac{1}{2} \right) = 0$$

$$0 \leq 2|\mathcal{F}| + 2|\mathcal{V}| - \frac{|\mathcal{D}|}{2}$$

$$|\mathcal{D}| \leq 4\left(|\mathcal{V}| + |\mathcal{F}|\right) \leq 4\left(M_{|\mathcal{V}|} + M_{|\mathcal{F}|}\right)$$

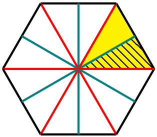


$$|\mathcal{D}| \leq \sum_{f \in \mathcal{F}} 2f$$

$$|\mathcal{D}| \leq 2 \left(\sum_{f \in \mathcal{R}_\mathcal{F}} f \right) + 2(M_{|\mathcal{F}|} - |\mathcal{R}_\mathcal{F}|) M_\mathcal{F}$$

Similiar:

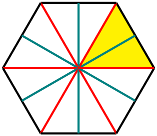
$$|\mathcal{D}| \leq 2 \left(\sum_{v \in \mathcal{R}_\mathcal{V}} v \right) + 2(M_{|\mathcal{V}|} - |\mathcal{R}_\mathcal{V}|) M_\mathcal{V}$$



$$\mathcal{C} : \mathbb{N} \to \mathbb{N}; n \mapsto \mathcal{C}(n) = \begin{cases} \frac{n}{6} & n \bmod 6 = 0 \\ \frac{n}{4} & n \bmod 4 = 0 \\ \frac{n}{3} & n \bmod 3 = 0 \\ \frac{n}{2} & n \bmod 2 = 0 \\ n & \text{all other cases} \end{cases}$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{F}} \mathcal{C}(f)$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{R}_\mathcal{F}} \mathcal{C}(f) + (m_{|\mathcal{F}|} - |\mathcal{R}_\mathcal{F}|) \min\{\mathcal{C}(n) | m_\mathcal{F} \leq n \leq M_\mathcal{F} \wedge n \notin \mathcal{U}_\mathcal{F}\}$$



Orientable tiling

$$\mathcal{C}^\circ : \mathbb{N} \to \mathbb{N}; n \mapsto \mathcal{C}^\circ(n) = \begin{cases} \frac{n}{3} & n \bmod 6 = 0 \\ \frac{n}{2} & n \bmod 4 = 0 \\ \frac{2n}{3} & n \bmod 3 = 0 \\ n & n \bmod 2 = 0 \\ 2n & \text{all other cases} \end{cases}$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{F}} \mathcal{C}^\circ(f)$$

$$|\mathcal{D}| \geq \sum_{f \in \mathcal{R}_\mathcal{F}} \mathcal{C}^\circ(f) + (m_{|\mathcal{F}|} - |\mathcal{R}_\mathcal{F}|) \min\{\mathcal{C}^\circ(n) | m_\mathcal{F} \leq n \leq M_\mathcal{F} \wedge n \notin \mathcal{U}_\mathcal{F}\}$$

Similar:

$$|\mathcal{D}| \geq \sum_{v \in \mathcal{R}_\mathcal{V}} \mathcal{C}(v) + (m_{|\mathcal{V}|} - |\mathcal{R}_\mathcal{V}|) \min\{\mathcal{C}(n) | m_\mathcal{V} \leq n \leq M_\mathcal{V} \wedge n \notin \mathcal{U}_\mathcal{V}\}$$

$$|\mathcal{D}| \geq \sum_{v \in \mathcal{R}_\mathcal{V}} \mathcal{C}^\circ(v) + (m_{|\mathcal{V}|} - |\mathcal{R}_\mathcal{V}|) \min\{\mathcal{C}^\circ(n) | m_\mathcal{V} \leq n \leq M_\mathcal{V} \wedge n \notin \mathcal{U}_\mathcal{V}\}$$

$$\sum_{d\in\mathcal{D}}\left(\frac{1}{M_{\mathcal{F}}}+\frac{1}{M_{\mathcal{V}}}-\frac{1}{2}\right)\leq\sum_{d\in\mathcal{D}}\left(\frac{1}{m_{01}(d)}+\frac{1}{m_{12}(d)}-\frac{1}{2}\right)\leq\sum_{d\in\mathcal{D}}\left(\frac{1}{m_{\mathcal{F}}}+\frac{1}{m_{\mathcal{V}}}-\frac{1}{2}\right),$$

$$0\leq 2m_{\mathcal{V}}+2m_{\mathcal{F}}-m_{\mathcal{V}}m_{\mathcal{F}}$$

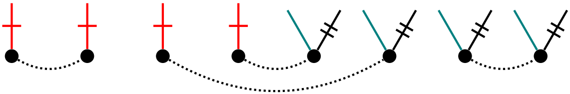
$$0\geq 2M_{\mathcal{V}}+2M_{\mathcal{F}}-M_{\mathcal{V}}M_{\mathcal{F}}$$

$$0 \leq 2m_{\mathcal{V}} + 2m_{\mathcal{F}} - m_{\mathcal{V}} m_{\mathcal{F}}$$

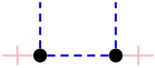
	3	4	5	6	7
3	3	2	1	0	-1
4	2	0	-2	-4	-6
5	1	-2	-5	-8	-11
6	0	-4	-8	-12	-16
7	-1	-6	-11	-16	-21

$$0 \geq 2M_{\mathcal{V}} + 2M_{\mathcal{F}} - M_{\mathcal{V}}M_{\mathcal{F}}$$

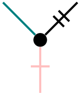
	3	4	5	6	7
3	3	2	1	0	-1
4	2	0	-2	-4	-6
5	1	-2	-5	-8	-11
6	0	-4	-8	-12	-16
7	-1	-6	-11	-16	-21



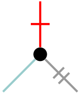
$$|\mathcal{V}| \geq \left\lceil \frac{S_1 + Q_4}{2} \right\rceil \qquad |\mathcal{F}| \geq \left\lceil \frac{S_1 + Q_4}{2} \right\rceil$$



$$2 \times \frac{1}{2}\,\langle\sigma_0,\sigma_1\rangle\text{-orbit or } 2 \times \frac{1}{2}\,\langle\sigma_1,\sigma_2\rangle\text{-orbit}$$



$$\frac{1}{2}\,\langle\sigma_0,\sigma_1\rangle\text{-orbit and } \frac{1}{2}\,\langle\sigma_1,\sigma_2\rangle\text{-orbit}$$



$$\frac{1}{2}\,\langle\sigma_0,\sigma_1\rangle\text{-orbit and } \frac{1}{2}\,\langle\sigma_1,\sigma_2\rangle\text{-orbit}$$

$$2\left\lceil \frac{S_1 + Q_4}{2} \right\rceil + Q_3 \leq |\mathcal{V}| + |\mathcal{F}|$$

Example

$$M_{|\mathcal{F}|} = 1$$

$$|\mathcal{D}| \in [1, 12]$$

$$|\mathcal{F}| = 1$$

$$|\mathcal{V}| \in [1, 6]$$

$$\mathcal{F} \subset [3, 6]$$

$$\mathcal{V} \subset [3, 26]$$

$$\mathcal{R}_{\mathcal{F}} = \mathcal{R}_{\mathcal{V}} = \mathcal{U}_{\mathcal{F}} = \mathcal{U}_{\mathcal{V}} = \emptyset$$

$\Rightarrow$ 93 Delaney-Dress symbols

Assigning the vertex and face numbers

Definitions

F = set of $\langle\sigma_0, \sigma_1\rangle$-orbits
V = set of $\langle\sigma_1, \sigma_2\rangle$-orbits

m_{01}, resp. m_{12} is constant on elements of F, resp. V.

$m^*_{01} : F \to \mathbb{N}; f \mapsto m^*_{01}(f) = m_{01}(d)$ with $d \in f$
$m^*_{12} : V \to \mathbb{N}; v \mapsto m^*_{12}(v) = m_{12}(d)$ with $d \in v$

$$0 = \sum_{f\in F} \frac{|f|}{m^*_{01}(f)} + \sum_{v\in V} \frac{|v|}{m^*_{12}(v)} - \frac{|\mathcal{D}|}{2}.$$

$$\sum_{v \in V} |v| = \sum_{f \in F} |f| = |\mathcal{D}|$$

$$\Downarrow$$

$$\sum_{f \in F} \frac{|f|}{m^*_{01}(f)} + \sum_{v \in V} \frac{|v|}{m^*_{12}(v)} = \sum_{f \in F} \frac{|f|}{4} + \sum_{v \in V} \frac{|v|}{4}$$

Current work

Try to determine the functions m^*_{01} and m^*_{12} in an efficient way.

Thank you for your attention.

Definitions

$$C := V \cup F$$

$$m : C \to \mathbb{N} : c \mapsto m(c) = \begin{cases} m^*_{01}(c) & c \in F \\ m^*_{12}(c) & c \in V \end{cases}$$

$$s : C \to \mathbb{N} : c \mapsto s(c) = |c|$$

$$r : C \to \mathbb{N}; c \mapsto r(c) = \begin{cases} s(c) & \text{if } c \text{ contains semi-edges} \\ \frac{s(c)}{2} & \text{in all other cases.} \end{cases}$$

1. $\displaystyle\sum_{c \in C} \frac{s(c)}{m(c)} = \sum_{c \in C} \frac{s(c)}{4}$
2. $\forall c \in C : r(c) \mid m(c)$
3. $\forall c \in C : m(c) \geq 3$

A necessary condition

Definitions

$n : C \to \mathbb{N}; c \mapsto n(c) = \frac{m(c)}{r(c)}$

$d : C \to \mathbb{N}; c \mapsto d(c) = \frac{s(c)}{r(c)}$

$$S := \sum_{c \in C} \frac{s(c)}{4} = \sum_{c \in C} \frac{s(c)}{m(c)} = \sum_{c \in C} \frac{d(c)}{n(c)} = \frac{|\mathcal{D}|}{2}$$

$C_1 := d^{-1}(1)$ and $C_2 := d^{-1}(2)$

$Im(d) = \{1, 2\} \Rightarrow C_1 \cup C_2 = C$

Theorem

If there exists a function n such that conditions 1 and 2 are satisfied, then $S \leq |C_1| + 2|C_2|$.